Subtrees with small branching number

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Definition

- Let T be a tree and $x \in T$. The set of *immediate successors* of T is denoted by $I_T(x)$.
- Let λ be a cardinal. A tree T is λ-*branching* [respectively $<\lambda$-*branching*] iff for every $x \in T$, $|I_T(x)| = \lambda$ [respectively $|I_T(x)| < \lambda$].
- A tree T is *finitely branching* if it is $<\aleph_0$-branching. It is *infinitely branching* if $|I_T(x)| \geq \aleph_0$ for every $x \in T$.
- A *subtree* of a tree T is a subset S of T such that for every $s \in S$ and $t \in T$, if $t < s$, then $t \in S$.

Lemma

Let T be an $\aleph_1$-tree. If T has no uncountable 1*-branching subtrees, then T is Aronszajn.*

Proof.

If b is a cofinal branch, then b is an uncountable 1-branching subtree. □

Definition

An infinitely branching $\aleph_1$-tree T is *Lindelöf* iff every finitely branching subtree of T is countable.

The previous lemma shows that

$$\{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

We will show that the inclusion is proper.
First, we explain the terminology: Trees are Lindelöf if and only if they have they are Lindelöf spaces with respect to a natural topology.

Definition (Nyikos)

Let T be a tree. The *fine wedge topology* on T is generated by all sets of the form $\uparrow x$ and their complements, where $\uparrow x = \{y \in T : x \leq y\}$ and $x \in T$.

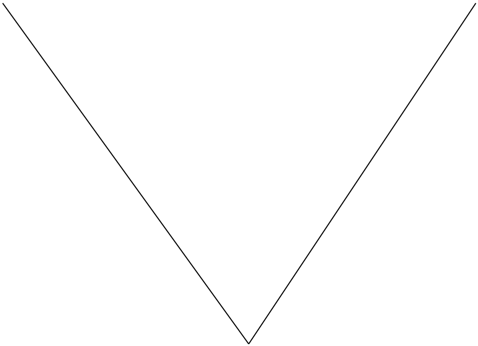
If $X \subseteq T$, write $\uparrow X = \{y \in T : \exists x \in X (x \leq y)\}$.

Lemma

If $x \in T$*, the family* $\{\uparrow x \setminus \uparrow F : F \in [I(x)]^{<\omega}\}$ *is a local basis of open neighbourhoods of* x*. In particular, the topology is Hausdorff.*

Proof.

Let $x \in W = (\uparrow x_0) \cap \cdots \cap (\uparrow x_{n-1}) \cap (\uparrow y_0)^c \cap \cdots \cap (\uparrow y_{m-1})^c$.



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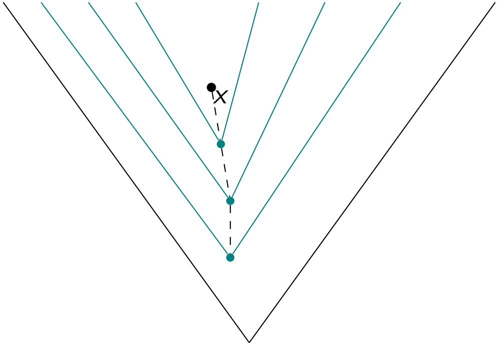
Since T is a tree, $\{x_i : i < n\}$ is a chain, say with maximum x_0.



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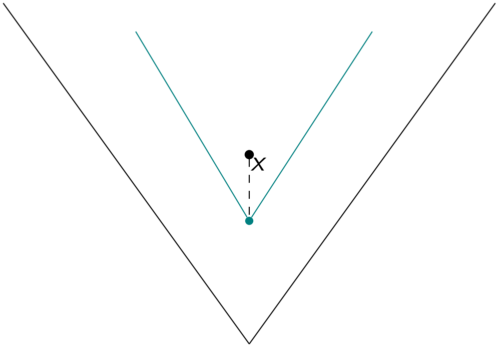

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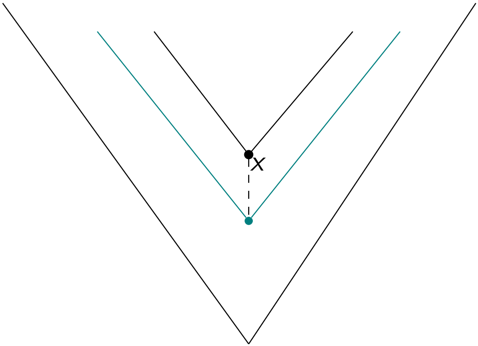

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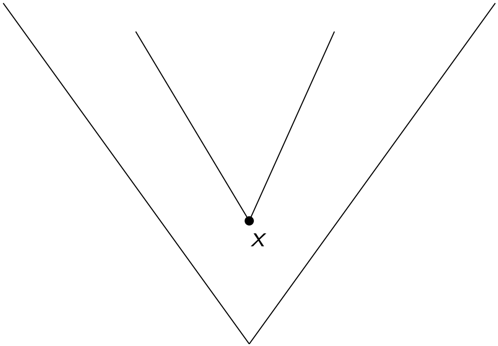

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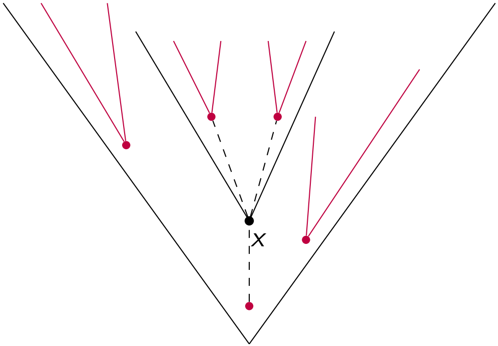

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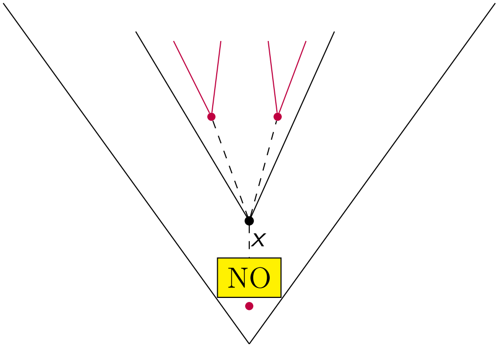

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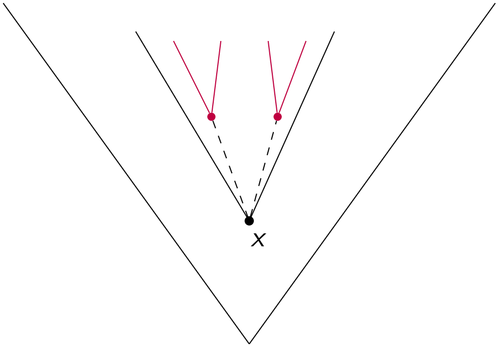

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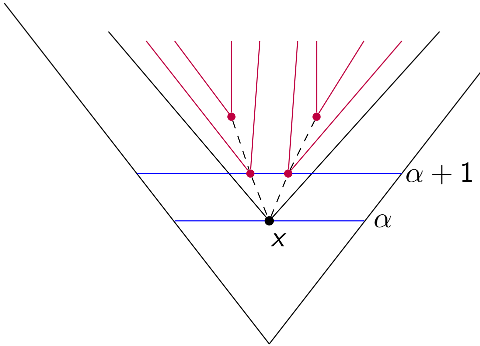


□

It therefore suffices to study open covers of the form
$\mathcal{U}_f = \{\uparrow x \setminus \uparrow f(x) : x \in T\}$, where $f \in \prod_{x \in T} [I(x)]^{<\omega}$.

Definition

Given $f \in \prod_{x \in T} [I(x)]^{<\omega}$, we will say that $x \in T$ is *safe* iff for every $y < x$, $x \in \uparrow f(y)$.

Note that the set of safe points is a finitely branching subtree of T.

Lemma

Let T be an infinitely branching $\aleph_1$-tree and $f \in \prod_{x \in T} [I(x)]^{<\omega}$. The following are equivalent:

1. *f has no countable subcover.*
2. *Every level has a safe point.*
3. *The set of $\{ht(x) : x \text{ is safe}\}$ is uncountable.*

Theorem (M.)

Let T be an infinitely branching $\aleph_1$-tree. Then T is a Lindelöf tree if and only if it has the Lindelöf property with respect to the fine wedge topology.

Proof.

$\Rightarrow$) Let f code a cover with no countable subcover. Let S be the set of safe points. By the previous lemma, S is uncountable.
$\Leftarrow$) Let S be an uncountable finitely branching subtree of T. Define $f(x) = \emptyset$ for $x \in T \setminus S$ and $f(x) = I_S(x)$ for $x \in S$. If $x \in S$ and $y < x$, then $x \in \uparrow f(y)$, so x is safe for S. So $S \subseteq \{x \in T : x \text{ is safe}\}$, hence the latter is uncountable. □

Recall that if T is a tree, an *antichain* is a set of pairwise incomparable elements of T. A *Suslin tree* is a tree with no uncountable chains or antichains.
If T is a tree, $\mathbb{P}_T$ is the dual order.

Lemma (folklore [2])

Let T be a Suslin tree. The poset $\mathbb{P}_T$ has the ccc and is countably distributive. Moreover, if $D \subseteq \mathbb{P}_T$ is dense and open, then there exists $\alpha < \omega_1$ such that $T \restriction [\alpha, \omega_1) \subseteq D$.

Lemma (folklore [2])

Let T be a Suslin tree in the universe V. If W is an outer model and $b \in W$ is a cofinal branch through T, then b is $\mathbb{P}_T$-generic over V.

Theorem (M.)

Let T be an infinitely branching Suslin tree. Then T is Lindelöf.

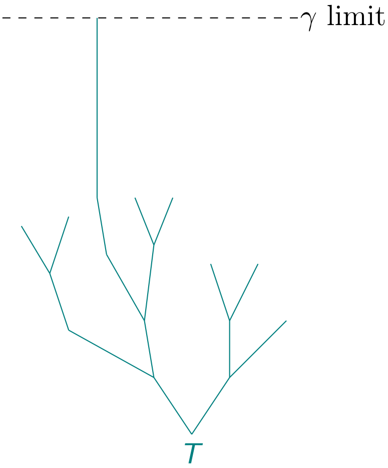
Proof.

Suppose not. Let $S \subseteq T$ be a finitely branching uncountable subtree. Then S is also Suslin. Force with $\mathbb{P}_S$ to add a branch b. By the previous lemma, b is $\mathbb{P}_T$-generic over V. But S is finitely branching and T is infinitely branching, so b is disjoint from S above some node of T, by a density argument. □

$$\{\text{Suslin}\} \subseteq \{\text{Lindelöf}\} \subseteq \{\text{Aronszajn}\}$$

A non-Lindelöf Aronszajn tree

Let $\vec{e} = \langle e_\alpha : \alpha < \omega_1 \rangle$ be a coherent sequence of injections, i.e. $e_\alpha : \alpha \to \omega$ is injective and $e_\alpha =^* e_\beta$.
Use a bijection $f : \omega_1 \times \omega \to \omega_1$ to transfer $\vec{e}$ to a coherent sequence $\langle f[e_\alpha] : \alpha < \omega_1 \rangle$, which is then used to build a binary Aronszajn tree T.
Now recursively build, level by level, an infinitely branching tree U which has T as a subtree.



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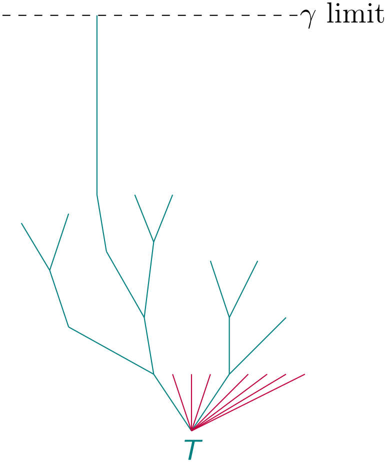

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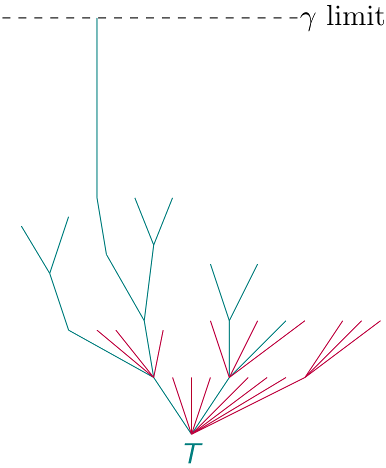

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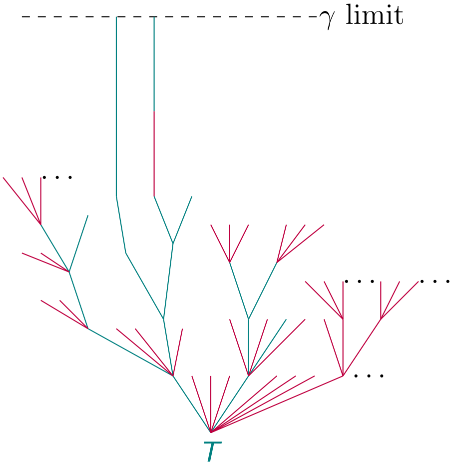


$\{\text{Suslin}\} \subseteq \{\text{Lindelöf}\} \subsetneq \{\text{Aronszajn}\}$

($\diamondsuit$) A special Lindelöf tree

Let $\vec{f} = \langle f_\alpha : \alpha \in \lim(\omega_1) \rangle$ be such $f_\alpha : \alpha \to [\alpha]^{<\omega}$ and $\vec{f}$ guesses any $f : \omega_1 \to [\omega_1]^{<\omega}$ stationarily often.
Build an infinitely branching tree T with underlying set ω_1 together with a specializing function $\varphi : T \to \mathbb{Q}$ by recursion on levels, maintaining that if $\varphi(x) < q$ then there is some $y > x$ with $\varphi(y) = q$.
At stage $\alpha = \omega \cdot \alpha$, for each pair $(x, q) \in (T \restriction \alpha) \times \mathbb{Q}$ with $\varphi(x) < q$, choose a cofinal branch b through $T \restriction \alpha$ such that $x \in b$, $\sup(\varphi``b) = q$ and the unique point of b immediately above x is not in $f_\alpha(x)$. Then put a new node $y \in T_\alpha$ above b and let $\varphi(y) = q$.
Every $y \in T_\alpha$ is obtained as above from some (x, q) and $y \in (\uparrow x) \setminus \uparrow f_\alpha(x)$, hence $f_\alpha \restriction (T \restriction \alpha)$ covers T.
Therefore, T is Lindelöf and so, under $\diamondsuit$,

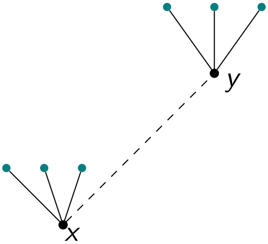
$$\{\text{Suslin}\} \subsetneq \{\text{Lindelöf}\} \subsetneq \{\text{Aronszajn}\}$$

Definition

Let T be a normal infinitely branching $\aleph_1$-tree. Consider the following poset $\mathbb{P}$: conditions are functions $p \in \prod_{x \in F}[I(x)]^{<\omega}$, where $F \in [T]^{<\omega}$, such that:

$$\forall x, y \in \mathrm{dom}(p)\,(x < y \to y{\restriction}(\mathrm{ht}(x)+1) \in p(x))\,.$$

The order on $\mathbb{P}$ is inclusion, $p \leq q \iff p \supseteq q$.



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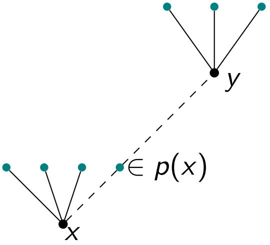

The generic subtree will be $\dot{S} = \bigcup \{\mathrm{dom}(p) : p \in \dot{G}\}$.
If $p \in \mathbb{P}$ and $x \in \mathrm{dom}(p)$, then p is a promise that $p(x) = I_{\dot{S}}(x)$.

Theorem (M.)

If T is a normal, infinitely branching Aronszajn tree, then $\mathbb{P}$ has the ccc and $\Vdash_{\mathbb{P}} \dot{S}$ is a finitely branching normal subtree of T.

Is T still Aronszajn in $V^{\mathbb{P}}$?

Corollary

Let $\mathbb{S}$ be Baumgartner's poset for specializing T with finite conditions. Then $\mathbb{S} \times \mathbb{P}$ is a ccc poset which forces that T is a special non-Lindelöf Aronszajn tree.

Corollary

If $\mathrm{MA}_{\aleph_1}$ holds, then there are no Lindelöf trees.

Other ways of adding subtrees?

Theorem (M.)

Let T be an infinitely branching $\aleph_1$ tree. Suppose $\mathbb{P}$ is a poset, G is $\mathbb{P}$-generic over V and $S \in V[G]$ is a finitely branching subtree of T. Suppose that $\mathbb{P}$ is either

- *countably closed*
- *strongly proper for a stationary set of countable elementary substructures of some (large) H_λ.*

Then $S \in V$.

Thank you :)

References

[1] Tomek Bartoszyński and Haim Judah. *Set Theory*. A K Peters, Ltd., Wellesley, MA, 1995. xii+546.

[2] Keith J. Devlin and Håvard Johnsbråten. *The Souslin Problem*. Lecture Notes in Mathematics 405. Springer, 1974. 132 pp.

[3] Pedro E. Marun. "Square Compactness and Lindelöf Trees". In: *In preparation* (2023).

[4] Peter J. Nyikos. "Various Topologies on Trees". In: *Proceedings of the Tennessee Topology Conference (Nashville, TN, 1996)*. 1997.