SQUARE COMPACTNESS AND LINDELÖF TREES

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Abstract. We prove that every weakly square compact cardinal is a strong limit cardinal, and therefore weakly compact. We also study Aronszajn trees with no uncountable finitely splitting subtrees, characterizing them in terms of being Lindelöf with respect to a particular topology. We prove that the class of such trees is consistently non-empty and lies between the classes of Suslin and Aronszajn trees.

1. Introduction

Recall that a topological space is *Lindelöf* if and only if every open cover has a countable subcover. Unlike compactness, the Lindelöf property need not be presserved by finite products, as shown by the classical Sorgenfrey line example, which is the space X with underlying set $\mathbb{R}$ and topology generated by all left-closed right-open intervals. This space is Lindelöf, but the uncountable set $\{(x,-x) : x \in \mathbb{R}\}$ is closed and discrete in X^2, hence X^2 is not Lindelöf. For details, see [10, Counterexample 84].

Extending this to larger cardinals κ, we say that a topological space X is κ-*compact* if and only if every open cover of X has a subcover of size less than κ. So, compact is $\aleph_0$-compact and Lindelöf is $\aleph_1$-compact. In connection with this, Hajnal and Juhász [3] introduced the following large cardinal notion: an infinite cardinal κ is *square compact* if and only if for every κ-compact space X, X^2 is κ-compact. This is in fact equivalent to the product of any two κ-compact spaces being κ-compact. For the non-trivial direction, if X and Y are κ-compact, then so is their disjoint sum $X \oplus Y$. By assumption, $(X \oplus Y)^2$ is κ-compact. Since $X \times Y$ is a closed subset of $(X \oplus Y)^2$, it follows that $X \times Y$ is κ-compact too.

Recall that the *weight* $w(X)$ of a topological space X is the least size of a base for the topology on X. A refined version of square compactness, graduated by weights, was introduced by Buhagiar and Džamonja in their 2021 paper [1]: given some cardinal λ, an infinite cardinal κ is λ-*square compact* if and only if for every space X of weight $\leq \lambda$, if X is κ-compact, then X^2 is κ-compact They (and we) say that κ is *weakly square compact* if and only if it is κ-square compact.

In this terminology, the results in [3] can be stated as:

Theorem (Hajnal-Juhász [3, Theorem 1])**.** *Every weakly square compact cardinal is regular.*

Theorem (Hajnal-Juhász [3, Theorem 2])**.** *Suppose that κ is uncountable. If κ is 2^κ-square compact, then it is weakly compact.*

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This shows that the existence of κ which is 2^κ-square compact is already a large cardinal notion. In [1], Buhagiar and Džamonja undertake a closer study of weak square compactness, and give a variety of equivalent formulations. In particular, they proved the following:

Theorem (Buhagiar-Džamonja, [1, Theorem 5.1])**.** *Let κ be an uncountable cardinal. Suppose that $\kappa^{<\kappa} = \kappa$. Then κ is weakly compact if and only if it is weakly square compact.*

Note that if κ is weakly compact, then $\kappa^{<\kappa} = \kappa$. The same is not obviously true if instead κ is κ-square compact. The first result we establish in this paper is that indeed $\kappa = \kappa^{<\kappa}$ whenever κ is κ-square compact, thereby removing the cardinal arithmetic assumption from the Buhagiar-Džamonja theorem. This is done by generalizing the Sorgenfrey line construction.

As far as we know, strong compactness continues to be the best upper bound for the consistency strength of full square compactness, see [5, Theorem 5.11], [8] and [1, Theorem 2.10]. Sharper upper bounds for the consistency strength of "There exists κ which is 2^κ-square compact" appear in [1, Theorem 6.1].

Having studied topologies on linear orders, we turn to looking at topologies on trees, with a view towards introducing new examples of Lindelöf spaces. In the survey [9], Nyikos considers a total of ten different topologies on trees. Of these, only two are always Hausdorff, and we adhere to the doctrine of only considering Hausdorff spaces. By [9, Theorem 3.6], the *coarse wedge topology* appears uninteresting for our purposes, since it is ω_1-compact[1] if and only if the underlying tree has countably many minimal elements. This leaves us with only the *fine wedge topology* to focus on. We give a tree-theoretic characterization of being Lindelöf with respect to this topology. First, some terminology: we say that a tree is *finitely splitting* if and only if every point in the tree has finitely many immediate successors. A tree is *infinitely splitting* if and only if every point in the tree has infinitely many immediate successors. A *subtree* of a tree T is a set $S \subseteq T$ such that for all $x \in S$ and $y \in T$, if $y < x$, then $y \in S$.

Theorem. *Let T be an infinitely splitting $\aleph_1$-tree. Then T is Lindelöf with respect to the fine-wedge topology if and only if every finitely splitting subtree of T is countable.*

We shall show that, for infinitely splitting trees,

$$\{\text{Suslin}\} \subseteq \{\text{Lindelöf}\} \subsetneq \{\text{Aronszajn}\}.$$

Here, by Lindelöf we mean Lindelöf with respect to the fine-wedge topology.

Given a partially ordered set X, we let X^* denote the *dual order* on X, that is $x <^* y$ if and only if $y < x$.

A *tree* is a pair $(T, <_T)$ such that $<_T$ is a strict partial order on T and $\{y \in T : y <_T x\}$ is well-ordered for every $x \in T$. We will usually suppress the subscript in $<_T$ and identify the tree with its underlying set when there is no danger of confusion.

Elements of a tree are referred to as *nodes* or *points*. We say $x, y \in T$ are *comparable*, denoted $x \parallel y$, if and only if $x \leq_T y$ or $y \leq_T x$. Otherwise, we say that x and y are *incomparable*, denoted $x \perp y$. The *height* of a node $x \in T$ is the order-type

[1] We caution the reader that, in Nyikos' survey, X being ω_1-compact is used to mean a different property than being Lindelöf.

of the set $\{y \in T : y <_T x\}$, denoted $\mathrm{ht}_T(x)$ (or simply $\mathrm{ht}(x)$). Given an ordinal α, *level* α of the tree is the set $T_\alpha = \{x \in T : \mathrm{ht}_T(x) = \alpha\}$. The *height* $h_T(T)$ of T is defined by $\mathrm{ht}(T) = \min\{\alpha : T_\alpha = \emptyset\}$. Given an ordinal $\alpha < \mathrm{ht}(T)$, we let $T\restriction\alpha = \{x \in T : \mathrm{ht}(x) < \alpha\}$, which is of course a subtree of T of height α. Given $x \in T$, we let $I_T(x)$ denote the set of *immediate successors* of x, and write $I(x)$ when there is no possibility of confusion.

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2. Square compactness

We shall say a space X is *hereditarily κ-compact* if and only if every subspace of X is κ-compact. For example, any space of weight less than κ is hereditarily κ-compact.

A useful criterion for hereditary κ-compactness is

Lemma 2.1. *Let (X, τ) be a topological space. Then X is hereditarily κ-compact if and only if for every $\mathcal{U} \subseteq \tau$ there is some $\mathcal{U}_0 \in [\mathcal{U}]^{<\kappa}$ with $\bigcup\mathcal{U} = \bigcup\mathcal{U}_0$.*

Proof. $\Rightarrow$) Given $\mathcal{U}$, consider the subspace $\bigcup\mathcal{U}$.

$\Leftarrow$) Suppose $Y \subseteq X$ is not κ-compact. Fix some $\mathcal{U} \subseteq \tau$ such that $\bigcup\mathcal{U} \supseteq Y$ but there is no $\mathcal{U}_0 \in [\mathcal{U}]^{<\kappa}$ with $\bigcup\mathcal{U}_0 \supseteq Y$. Then $\bigcup\mathcal{U}_0 \neq \bigcup\mathcal{U}$ for every $\mathcal{U}_0 \in [\mathcal{U}]^{<\kappa}$. □

The following is obvious:

Lemma 2.2. *Suppose that (X, τ) is κ-compact and $Y \subseteq X$ is closed. Then Y is κ-compact with the subspace topology.*

As mentioned in the introduction, Hajnal and Juhász already proved that weak square compactness entails regularity. To deal with the (strong) inaccessibility of κ, we will generalize the classical construction of the Sorgenfrey line to larger linear orders.

Definition 2.3. Let $(X, <)$ be a *dlo* (dense linear order without end-points). The *density* of X, denoted $d(X)$, is the cardinal

$$d(X) = \min\{|D| : D \text{ is dense in } X\}$$

This of course coincides with the density of X as a topological spacer under the order topology. ⌟

For example, $d(\mathbb{R}) = \aleph_0$. It is straightforward to show that $w(X)$, the weight of X with respect to the order topology, is exactly $d(X)$.

Definition 2.4. Given a dlo $(X, <)$, the family $\{[x, y) : x, y \in X \wedge x < y\}$ forms a base for a topology on X, which we shall call the *Sorgenfrey* topology. ⌟

Lemma 2.5. *Let $(X, <)$ be a dlo with $d(X) < \kappa$. Then the Sorgenfrey topology on X is hereditarily κ-compact.*

Proof. Let $\mathcal{U} \subseteq \{[x, y) : x, y \in X\}$ and let $W = \bigcup\{(x, y) : [x, y) \in \mathcal{U}\}$. Obviously, W is open with respect to the order topology on X, which has weight less than κ. By Lemma 2.1, $W = \bigcup\{(x, y) : [x, y) \in \mathcal{U}_0\}$ for some $\mathcal{U}_0 \in [\mathcal{U}]^{<\kappa}$. Let $A := (\bigcup\mathcal{U}) \setminus W$.

Claim. $|A| < \kappa$.

Proof of claim. Fix $D \in [X]^{<\kappa}$ dense in the order topology. For each $x \in A$, find $[a_x, b_x) \in \mathcal{U}$ such that $x \in [a_x, b_x)$. Since $x \notin W$, we infer that $x = a_x < b_x$, so we can pick some $d_x \in D$ with $x < d_x < b_x$. Now suppose $x, y \in A$ with $x < y$. Since $y \notin W$, $b_x \leq y$. Then $d_x < b_x \leq y < d_y$, so $d_x < d_y$. Therefore, $x \mapsto d_x$ is an injective map from A into D. ⊣

For each $x \in A$, pick $U_x \in \mathcal{U}$ with $x \in U_x$. Let $\mathcal{U}_1 = \{U_x : x \in A\}$. Clearly, $|\mathcal{U}_1| < \kappa$. We now have that $\mathcal{U}_2 = \mathcal{U}_0 \cup \mathcal{U}_1 \in [\mathcal{U}]^{<\kappa}$ and $\bigcup \mathcal{U}_2 = \bigcup \mathcal{U}$. □

Lemma 2.6. *Let $\kappa > \omega$ be a cardinal. Suppose that there is a dlo $(X, <)$ with $d(X) < \kappa = |X|$. Then κ is not κ-square compact.*

Proof. Replacing X by $X \oplus X^*$ if necessary, we may assume that $(X, <)$ admits an order reversing involution, which we shall suggestively denote by $x \mapsto -x$.

Let τ be the Sorgenfrey topology on X. Note that $w(X, \tau) \leq \kappa$, because $|X| \leq \kappa$, and that (X, τ) is κ-compact by lemma 2.5. It therefore suffices to show that X^2 is not κ-compact with respect to the product topology. Let

$$Y = \{(x, -x) : x \in X\}.$$

Since $x \mapsto -x$ is order-reversing, it is continuous with respect to the order topology $\tau_<$, hence Y is closed in $(X^2, \tau_< \otimes \tau_<)$. But $\tau_< \subseteq \tau$, so Y is closed in $(X^2, \tau \otimes \tau)$. For each $x \in X$, pick $u_x, v_x \in X$ with $u_x < x < v_x$. Now observe that

$$([x, v_x) \times [-x, -u_x)) \cap Y = \{(x, -x)\}.$$

We have shown that Y is discrete in $(X^2, \tau \otimes \tau)$. Since $|Y| = \kappa$, Y is not κ-compact, and so neither is $(X^2, \tau \otimes \tau)$ because Y is closed. □

The goal now is to build large dlo's with small density. This will be possible, under certain cardinal arithmetic constraints. Our original construction was rather convoluted, and we thank Will Brian for suggesting the following simpler approach.

Lemma 2.7. *Let $\kappa \geq \omega_1$. Suppose there exist infinite cardinals μ and θ such that $\mu^{<\theta} = \mu < \kappa \leq \mu^\theta$. Then there is a dlo X with $d(X) < \kappa = |X|$.*

Proof. Let $Y := {}^\theta\mu$, ordered lexicographically. Note that $|Y| = \mu^\theta \geq \kappa$. Let D be the set of sequences in Y which are eventually 0. Then D is dense in Y and $|D| = \mu^{<\theta} < \kappa$. By the Downward Lowenheim-Skölem theorem, find $X \prec Y$ with $D \subseteq X$ and $|X| = \kappa$. Since D is dense in X, $d(X) < \kappa$. □

Theorem 2.8. *Let $\kappa \geq \omega_1$. If there are cardinals μ and θ such that $\mu^{<\theta} = \mu < \kappa \leq \mu^\theta$, then κ is not κ-square compact.*

Proof. Immediate from Lemmas 2.6 and 2.7. □

Corollary 2.9. *Suppose $\lambda \geq \omega$. Then λ^+ is not λ^+-square compact.*

Proof. Let $\theta := \min\{\nu : \lambda^\nu > \lambda\}$. By König's lemma, $\theta \leq \mathrm{cf}(\lambda)$, so $\lambda^{<\theta} = \lambda$ by the minimality of θ. Now apply Lemma 2.7 with $\kappa = \lambda^+$ and $\mu = \lambda$. □

In particular, if κ is κ-square compact, then κ is a limit cardinal, hence weakly inaccessible. In fact, this can be improved:

Corollary 2.10. *Suppose κ is κ-square compact. Then κ is strongly inaccessible.*

Proof. The fact that κ is regular under the hypothesis was already mentioned in the introduction, and follows from [3, Theorem 1].

Suppose κ is not strong limit. Let

$$\theta = \min\{\nu : \exists\lambda\,(\nu \leq \lambda < \kappa \leq \lambda^\nu)\}.$$

To see that this is well defined, fix $\delta < \kappa$ so that $2^\delta \geq \kappa$, and take $\lambda = \nu = \delta$.

Having fixed θ, let $\lambda < \kappa$ be the least witness to the definition of θ, that is $\theta \leq \lambda < \kappa \leq \lambda^\theta$ and λ is least with these properties. Note that, if $\alpha < \theta$, then $\lambda^\alpha < \kappa$, since otherwise α contradicts the minimal choice of θ.

Claim 1. *θ is regular.*

Proof of claim 1. Suppose not, say $\theta^* = \mathrm{cf}(\theta) < \theta$. Fix $\langle\theta_\xi : \xi < \theta^*\rangle$ cofinal in θ. By the minimality of θ, $\lambda^{\theta_\xi} < \kappa$ for every $\xi < \theta^*$. Let $\lambda^* := \sup\{\lambda^{\theta_\xi} : \xi < \theta^*\}$. Since κ is regular and $\theta^* < \theta < \kappa$, it follows that $\lambda^* < \kappa$. We therefore have

$$\kappa \leq \lambda^\theta = \prod_{\xi<\theta^*} \lambda^{\theta_\xi} \leq (\lambda^*)^{\theta^*}$$

This contradicts the minimality of θ. ⊣

Put $\mu = \lambda^{<\theta}$. Again, $\mu < \kappa$, because $\lambda^\alpha < \kappa$ for $\alpha < \theta$ and $\theta < \kappa = \mathrm{cf}(\kappa)$. Also, $\mu^\theta \geq \lambda^\theta \geq \kappa$.

Claim 2. $\mu^{<\theta} = \mu$.

Proof of claim 2. We consider two separate cases.

Case 1: $\alpha \mapsto \lambda^\alpha$ is eventually constant for $\alpha < \theta$. Note that this includes the case when θ is a successor cardinal. By definition of μ, the eventual constant value must be μ, so $\lambda^\alpha = \mu$ for all large enough $\alpha < \theta$. But then $\mu^\alpha = \mu$ whenever $\alpha < \theta$ is sufficiently big, hence $\mu^{<\theta} = \mu$.

Case 2: λ^α is not eventually constant for $\alpha < \theta$. As θ is regular, $\mathrm{cf}(\mu) = \theta$. So, if $\alpha < \theta$, we have

$$\mu^\alpha = \sum_{\beta<\theta} (\lambda^\beta)^\alpha = \mu.$$

Therefore, $\mu^{<\theta} = \mu$. ⊣

Therefore, μ, θ and κ satisfy the conditions of Theorem 2.7, so κ is not κ-square compact. □

Theorem 2.11. *Let κ be an uncountable cardinal. Then κ is weakly compact if and only if it is κ-square compact.*

Proof. The forwards direction can be found in [3, Theorem 2]. The backwards direction is in [1, Theorem 5.1], under the additional hypothesis that $\kappa^{<\kappa} = \kappa$. But this is redundant when κ is κ-square compact, because κ is strongly inaccessible by Corollary 2.10. □

3. The fine wedge topology

Let $(T, <)$ be a tree. If $X \subseteq T$, we let $\uparrow X := \{y \in T : \exists x \in X(x \leq y)\}$. If $X = \{x\}$, we write $\uparrow x$ instead of $\uparrow\{x\}$. The symbols $\downarrow X$ and $\downarrow x$ are defined analogously.

If T is a tree, the *fine wedge topology* on T is generated by the sets $\uparrow t$ and their complements, where $t \in T$.

Note that, if $x < y$, then $(\uparrow x) \setminus \uparrow y$ and $\uparrow y$ are disjoint open neighbourhoods of x and y, respectively. If $x \perp y$, then $\uparrow x$ and $\uparrow y$ are disjoint open neighbourhoods of x and y. We have thus shown that the topology is Hausdorff.

All topological notions below refer to the fine wedge topology.

Remark 3.1. If T is *finitely splitting at* x (that is $|I(x)| < \aleph_0$), then the identity

$$\{x\} = (\uparrow x) \cap \bigcap_{y \in I(x)} (\uparrow y)^c$$

shows that x is isolated. Therefore, if T is finitely splitting, the fine wedge topology is just the discrete topology on T. The interplay between finite and infinitely splitting trees will play a key role in our work, see Theorem 3.8. ⌟

Recall that, if X is a topological space and $x \in X$, we say that a collection of open sets $\mathcal{B}$ is a *neighbourhood base at* x if and only if for every open set U with $x \in U$ there is some $B \in \mathcal{B}$ with $x \in B \subseteq U$. We define the *character of* x to be the cardinal $\chi(x, X) := \min\{|\mathcal{B}| : \mathcal{B} \text{ is a neighbourhood base at } x\}$.

Lemma 3.2. *Let T be a tree. Given $x \in T$, the sets*

$$(\uparrow x) \setminus \uparrow F,$$

where $F \in [I(x)]^{<\omega}$, form a neighbourhood base at x, and so $\chi(x, T) = |I(x)|$. In particular, if every node has $\aleph_0$ many immediate successors, then the fine-wedge topology is first countable.

Proof. Let U be a basic open neighbourhood of x, say

$$x \in U = \bigcap_{i<n} \uparrow x_i \setminus \bigcup_{j<m} \uparrow y_j$$

for some $x_i, y_j \in T$, $n, m \in \omega$. Let $J = \{j < m : x < y_j\}$. For each $j \in J$, let $z_j \in I(x)$ be the unique point with $z_j \leq y_j$. Then

$$x \in (\uparrow x) \setminus \bigcup_{j \in J} \uparrow z_j \subseteq U,$$

which completes the proof. □

We isolate the following elementary result from general topology, whose proof is easy and hence omitted:

Lemma 3.3. *Let X be a topological space and κ an infinite cardinal. Suppose we have a sequence $\langle \mathcal{B}_x : x \in X \rangle$ such that $\mathcal{B}_x$ is a neighbourhood base at x for every $x \in X$. Then X is κ-compact if and only if for each $\Gamma \in \prod_{x \in X} \mathcal{B}_x$ there is some $Y \in [X]^{<\kappa}$ such that $X = \bigcup_{y \in Y} \Gamma(y)$.*

In the tree context, we'll be looking at the system of neighbourhood bases formed by the sets $\uparrow x \setminus \uparrow f(x)$, where $f \in \prod_{x \in T} [I(x)]^{<\omega}$. We shall say that such an f *codes* the cover $\mathcal{U}_f := \{\uparrow x \setminus \uparrow f(x) : x \in T\}$. Going forward, we will blur the distinction between the function f and the open cover $\mathcal{U}_f$, and speak simply of the *cover* f. We will also consider $\mathcal{U}_f$ for $f \in \prod_{x \in X} [I(x)]^{<\omega}$, where $X \subseteq T$ (of course, $\mathcal{U}_f$ might not cover T).

Note that covers of this kind have the following important property:

Lemma 3.4. *Let T be a tree and $f \in \prod_{x \in T} [I(x)]^{<\omega}$. The following are equivalent:*

(1) *f has a countable subcover.*

(2) *There is a limit ordinal* $\alpha < \omega_1$ *such that* $f\restriction(T\restriction\alpha)$ *covers* T.
(3) *There is an ordinal* $\alpha < \omega_1$ *such that for every* $x \in T_\alpha$ *there is some* $y \in T\restriction\alpha$ *such that* $x \in \uparrow y \setminus \uparrow f(y)$.

Proof. Trivial. □

Definition 3.5. Let T be an $\aleph_1$-tree and $f \in \prod_{x\in T}[I(x)]^{<\omega}$. We say that a point $x \in T$ is *safe* (for f) if and only if for all $y < x$, $x \in \uparrow f(y)$. ⌟

An immediate consequence of the definition is:

Lemma 3.6. *Let* T *be an* $\aleph_1$*-tree and* $f \in \prod_{x\in T}[I(x)]^{<\omega}$. *If* $x \in T$ *is safe, then so is every* $y < x$. *Also, if* $z \in I(x)$ *(with* x *safe), then* z *is safe if and only if* $z \in f(x)$.

Proof. Trivial. □

The key property of safe points is the following:

Lemma 3.7. *Let* T *be an infinitely splitting* $\aleph_1$*-tree and* $f \in \prod_{x\in T}[I(x)]^{<\omega}$. *The following are equivalent:*

(1) f *has no countable subcover.*
(2) *For every* $\alpha < \omega_1$ *there is a safe point of height* α.
(3) *The set* $\{\mathrm{ht}(x) : x \text{ is safe}\}$ *is unbounded in* ω_1.

Proof. (1) ⇒ (2): Fix $\alpha < \omega_1$. By Lemma 3.4, there is some $x \in T_\alpha$ such that for all $y \in T\restriction\alpha$, $x \notin \uparrow y \setminus \uparrow f(y)$. In particular, if $y < x$, $x \in \uparrow f(y)$, so x is safe.

(2) ⇒ (3): Trivial.

(3) ⇒ (1): Suppose towards a contradiction that f has a countable subcover. By Lemma 3.4, there is some limit $\gamma < \omega_1$ such that $f\restriction(T\restriction\gamma)$ covers T. Choose a safe point x with $\gamma < \mathrm{ht}(x)$. Let $y \in T\restriction\gamma$. If $y \not< x$, then obviously $x \notin \uparrow y \setminus \uparrow f(y)$. If $y < x$, the safety of x implies that $x \in \uparrow f(y)$, so $x \notin \uparrow y \setminus \uparrow f(y)$. In either case, $x \notin \bigcup \mathcal{U}_{f\restriction(T\restriction\gamma)} = T$, contradiction. □

Recall that a subtree of a tree is a downwards closed subset. Note that, if S is a subtree of T and $\alpha < \mathrm{ht}(T)$, then $S_\alpha = S \cap T_\alpha$. Also, if $x \in S$, then $I_S(x) = I_T(x) \cap S$.

Theorem 3.8. *Let* T *be an infinitely splitting* $\aleph_1$*-tree. Then* T *is Lindelöf if and only if every finitely splitting subtree of* T *is countable.*

Proof. ⇒) Let $S \subseteq T$ be a finitely branching subtree of T. Note that the subspace topology on S is just the fine wedge topology of S as a tree. Since S is finitely splitting, it is closed and discrete (see Remark 3.1), hence T cannot be Lindelöf.

⇐) Let f code a cover with no countable subcover. Let S be the set of points of S which are safe for f. It is clear that S is a subtree of T. Given $x \in S$, we see that $I_S(x) = f(x)$, so S is finitely splitting. Since f has no countable subcover, $\mathrm{ht}(S) = \aleph_1$. □

Corollary 3.9. *Every Lindelöf tree is Aronszajn.*

Proof. Let T be a Lindelöf tree and $b \subseteq T$ a branch. Then $\downarrow b$ is a finitely splitting subtree of T, hence countable by Theorem 3.8. □

Remark 3.10. Recall that, if there is a Suslin tree, then there is one that is infinitely splitting, see [4, Lemma 9.13]. ⌟

Theorem 3.11. *Let T be an infinitely splitting Suslin tree. Then T is Lindeöf.*

Proof. Suppose that S is a finitely splitting subtree of T with height $\aleph_1$. Fix a sequence $\langle x_\alpha : \alpha < \omega_1 \rangle$ with $x_\alpha \in S_\alpha$. For each $\alpha < \omega_1$, choose $y_\alpha \in I_T(x_\alpha) \setminus S$. We show that $\{y_\alpha : \alpha < \omega_1\}$ is an antichain. Indeed, if $\alpha < \beta$ and $y_\alpha < y_\beta$, then $y_\alpha \leq x_\beta$ because $y_\beta \in I_T(x_\beta)$. But then $y_\alpha \in S$, contradiction.

We have shown that every finitely splitting subtree of T is countable. By 3.8, T is Lindelöf. □

4. Examples of Lindelöf and non-Lindelöf trees

If s and t are two functions with domain α, we let $\Delta(s,t) := \{\xi < \alpha : s(\xi) \neq t(\xi)\}$. We write $s =^* t$ if and only if $\Delta(s,t)$ is finite.

We shall say that $\langle e_\alpha : \alpha < \omega_1 \rangle$ is a *coherent sequence of injections* if $e_\alpha : \alpha \to \omega$ is injective and $e_\alpha =^* e_\beta \restriction \alpha$ for all $\alpha, \beta < \omega_1$. Note that each e_α must have coinfinite range. Going forward, we shall speak simply of coherent sequences, which in the literature usually refers to finite to one functions. Let

$$T^{\vec{e}} = \bigcup_{\alpha < \omega_1} \{s \in {}^\alpha\omega : s \text{ is injective} \wedge s =^* e_\alpha\}.$$

It is clear that $T^{\vec{e}}$ is an infinitely splitting Aronszajn tree.

Our next goal is to construct a non-Lindelöf Aronszajn tree. The idea is to build a finitely splitting Aronszajn tree, and then make $\aleph_0$ many new nodes "sprout" at each node, producing a larger, infinitely splitting tree. The constructions of Aronszajn trees that we're familiar with all produce infinitely splitting trees, so our first task is to build a finitely splitting one.

Recall that a tree if *splitting* if every node has at least two distinct immediate successors.

Lemma 4.1. *There is a splitting subtree of $2^{<\omega_1}$ which is Aronszajn.*

Proof. Fix a coherent sequence $\vec{e}$ and a bijection $f : \omega_1 \times \omega \to \omega_1$ such that for every limit ordinal $\gamma < \omega_1$, $f[\gamma \times \omega] = \gamma$. Put $\Gamma = \lim(\omega_1) \cup \{0\}$. Define $x_\alpha \in 2^\alpha$ for $\alpha \in \Gamma$ by $x_\alpha = \chi_{f[e_\alpha]}$, where χ_A denotes the characteristic function of A. This makes sense because $f[e_\alpha] \subseteq \alpha$.

Claim. *If $\alpha, \beta \in \Gamma$ and $\alpha < \beta$ then $x_\alpha =^* x_\beta \restriction \alpha$.*

Proof of claim. For readability, we extend our $\Delta(s,t)$ notation to allow functions with different domains. More precisely, if $\mathrm{dom}(s) \leq \mathrm{dom}(t)$, we write $\Delta(s,t)$ for $\Delta(s, t \restriction \mathrm{dom}(s))$.

If $\alpha = 0$ everything is trivial so we assume that $\alpha \geq \omega$. Fix $\eta \in \Delta(x_\alpha, x_\beta)$, so $x_\alpha(\eta) \neq x_\beta(\eta)$. Since $f[\alpha \times \omega] = \alpha$, there exist unique $\xi < \alpha$ and $n \in \omega$ with $f(\xi, n) = \eta$.

Since $x_\alpha(\eta) \neq x_\beta(\eta)$, there are two possibilities: $\eta \in f[e_\alpha] \setminus f[e_\beta]$ or $\eta \in f[e_\beta] \setminus f[e_\alpha]$. We consider the two cases separately:

1. Suppose that $\eta \in f[e_\alpha] \setminus f[e_\beta]$. Then $e_\alpha(\xi) = n$ but $e_\beta(\xi) \neq n$, so $\xi \in \Delta(e_\alpha, e_\beta)$ and $\eta = f(\xi, e_\alpha(\xi))$.
2. Suppose that $\eta \in f[e_\beta] \setminus f[e_\alpha]$. Then $e_\beta(\xi) = n$ but $e_\alpha(\xi) \neq n$, so $\xi \in \Delta(e_\alpha, e_\beta)$ and $\eta = f(\xi, e_\beta(\xi))$.

We have therefore established that

$$\Delta(x_\alpha, x_\beta) \subseteq \{f(\xi, e_\alpha(\xi)) : \xi \in \Delta(e_\alpha, e_\beta)\} \cup \{f(\xi, e_\beta(\xi)) : \xi \in \Delta(e_\alpha, e_\beta)\}.$$

Since $\vec{e}$ is coherent, the union on the right is finite, hence $\Delta(x_\alpha, x_\beta)$ is finite too. ⊣

Given $\alpha < \omega_1$, let γ_α be the unique ordinal in Γ such that $\gamma_\alpha \leq \alpha < \gamma_\alpha + \omega$. Note that $\alpha \in \Gamma$ if and only if $\alpha = \gamma_\alpha$. Define

$$T = \bigcup_{\alpha<\omega_1} \{x \in 2^\alpha : x{\restriction}\gamma_\alpha =^* x_{\gamma_\alpha}\}$$

By the claim, T is a subtree of $2^{<\omega_1}$. Indeed, if $x \in 2^\alpha$ and $y \in T \cap 2^\beta$ satisfy $x \subseteq y$, then $\gamma_\alpha \leq \gamma_\beta$, so $x{\restriction}\gamma_\alpha = (y{\restriction}\gamma_\beta){\restriction}\gamma_\alpha =^* x_{\gamma_\beta}{\restriction}\gamma_\alpha =^* x_{\gamma_\alpha}$, where the last $=^*$ follows from the claim. The fact that T has countable levels is immediate from the claim. As $x_\alpha \in T_\alpha$ for every $\alpha \in \Gamma$, we see that T has height ω_1, and so T is an $\aleph_1$-tree.

We point out that that, if $x \in T$, then $x^\frown\langle 0\rangle, x^\frown\langle 1\rangle \in T$, because $\gamma_\alpha = \gamma_{\alpha+1}$ for every $\alpha < \omega_1$.

To see that T is Aronszajn, assume towards a contradiction that $\langle y_\alpha : \alpha < \omega_1\rangle$ is a branch through T, so in particular $y_\alpha =^* x_\alpha$ for every $\alpha \in \lim(\omega_1)$. Find $\eta_\alpha < \alpha$ for each $\alpha \in \lim(\omega_1)$ so that $\Delta(y_\alpha, x_\alpha) \subseteq \eta_\alpha$. Since $\alpha \mapsto \eta_\alpha$ is regressive, by Fodor's Lemma there is some stationary set $E \subseteq \lim(\omega_1)$ and some $\eta < \omega_1$ such that $\eta_\alpha = \eta$ for every $\alpha \in E$. Since $|T_\eta| \leq \aleph_0$, we may find $E' \subseteq E$ stationary and $x, y \in 2^\eta$ such that $y_\alpha{\restriction}\eta = y$ and $x_\alpha{\restriction}\eta = x$ for every $\alpha \in E'$. If $\alpha, \beta \in E'$, $\alpha < \beta$,

$$x_\alpha{\restriction}[\eta, \alpha) = y_\alpha{\restriction}[\eta, \alpha) = y_\beta{\restriction}[\eta, \alpha) = x_\beta{\restriction}[\eta, \alpha)$$

by the choice of η. Since $x_\alpha{\restriction}\eta = x = x_\beta{\restriction}\eta$, it follows that $x_\alpha = x_\beta{\restriction}\alpha$. Finally, if $\xi < \alpha$ and $n := e_\alpha(\xi)$, then $x_\alpha(f(\xi, n)) = 1$, so $x_\beta(f(\xi, n)) = 1$, so $f(\xi, n) \in f[e_\beta]$, so $e_\beta(\xi) = n$. We have thus shown that $\langle e_\alpha : \alpha \in E'\rangle$ is a chain, which is absurd because E' is uncountable. □

Lemma 4.2. *There is an infinitely splitting Aronszajn tree with a finitely splitting subtree of height* $\aleph_1$.

Proof. Let $\vec{e}$ be a coherent sequence and let $T \subseteq 2^{<\omega_1}$ be the tree constructed from $\vec{e}$ in Lemma 4.1. We recursively define a tree $U \subseteq \omega^{<\omega_1}$ level by level, starting with $U_0 = \{\emptyset\}$. For successor stages, we let $U_{\alpha+1} = \{u^\frown\langle n\rangle : u \in U_\alpha \wedge n \in \omega\}$. If $\alpha < \omega_1$ is a limit ordinal, we let $U_\alpha = \{u \cup t{\restriction}[\mathrm{dom}(u), \alpha) : u \in U{\restriction}\alpha \wedge t \in T_\alpha\}$. An easy induction shows that U_α is countable and that $T_\alpha \subseteq U_\alpha$ for every α.

Claim 1. *If $\beta < \alpha$, $t \in T_\alpha$ and $u \in U_\beta$, then $u \cup t{\restriction}[\beta, \alpha) \in U_\alpha$.*

Proof of claim 1. By induction on α:

- If $\alpha = 0$, then it's obvious.
- Suppose that α is a successor ordinal, say $\alpha = \gamma + 1$. Then $u \cup (t{\restriction}[\beta, \alpha)) = (u \cup t{\restriction}[\beta, \gamma))^\frown\langle t(\gamma)\rangle$ which belongs to U_α by the inductive hypothesis and the definition of U_α.
- If α is a limit ordinal, this is immediate from the definition of U_α. ⊣

Claim 2. *If $\alpha < \omega_1$, $\beta < \alpha$ and $v \in U_\alpha$, then $v{\restriction}\beta \in U$.*

Proof of claim 2. By induction on α:

- If $\alpha = 0$ then it's vacuously true.

- Suppose that α is a successor ordinal, say $\alpha = \gamma + 1$. Fix $v \in U_\alpha$, so $v = u^\frown\langle n\rangle$ for some $u \in U_\gamma$ and $n \in \omega$. Let $\beta < \alpha$. If $\beta = \gamma$, then $v\restriction\beta = u \in U$ as desired. If $\beta < \gamma$, then $v\restriction\beta = u\restriction\beta \in U$ by the inductive hypothesis applied to γ.
- Suppose that α is a limit ordinal and let $v \in U_\alpha$, say $v = u\cup(t\restriction[\mathrm{dom}(u), \alpha))$, where $u \in U_\gamma$, $\gamma < \alpha$, and $t \in T$. Let $\beta < \alpha$. If $\beta = \gamma$, then $v\restriction\beta = u \in U_\beta$. If $\beta < \gamma$, then $v\restriction\beta = u\restriction\beta \in U$ by the inductive hypothesis applied to γ. If $\gamma < \beta$, then $v\restriction\beta = u \cup (t\restriction[\mathrm{dom}(u), \beta))$. We now induct on β. If β is a limit ordinal, then $v\restriction\beta \in U_\beta$ by definition of U_β. If β is a successor ordinal, say $\beta = \delta + 1$, then $v\restriction\beta = (u \cup t\restriction[\mathrm{dom}(u), \delta))^\frown\langle t(\delta)\rangle$. By Claim 1, $u \cup t\restriction[\mathrm{dom}(u), \delta) \in U_\delta$, and so $v\restriction\beta \in U_\beta$ by definition of U_β. $\dashv$

To see that U is Aronszajn, suppose towards a contradiction that b is a cofinal branch through U. Let x_α be the α^{th} point of b. For each limit ordinal $\alpha < \omega_1$, there exist $u_\alpha < x_\alpha$ and $t_\alpha \in T_\alpha$ such that $x_\alpha = u_\alpha \cup t_\alpha\restriction[\mathrm{dom}(u_\alpha), \alpha)$. Define $f : \lim(\omega_1) \to \omega_1$ by $f(\alpha) = \mathrm{dom}(u_\alpha)$. Since f is regressive, we may find a stationary set $\Gamma \subseteq \lim(\omega_1)$ and some $\eta < \omega_1$ such that $f``\Gamma = \{\eta\}$. As $|T_\eta| \leq \aleph_0$, we may assume, by shrinking Γ if necessary, that there is some $t \in T_\eta$ such that $t = t_\alpha\restriction\eta$ for all $\alpha \in \Gamma$. If $\alpha, \beta \in \Gamma$, $\alpha < \beta$, then $t_\alpha\restriction[\eta, \alpha) = x_\alpha\restriction[\eta, \alpha) = x_\beta\restriction[\eta, \alpha) = t_\beta\restriction[\eta, \alpha)$ by our choice of η. But then $t_\alpha = t_\beta\restriction\alpha$ because they both agree with t below η. This means that $\langle t_\alpha : \alpha \in \Gamma\rangle$ is an uncountable chain in T, contradicting that T is Aronszajn. $\square$

Corollary 4.3. *There is a non-Lindelöf Aronszajn tree.*

Proof. Apply lemmas 4.1 and 4.2 to obtain an infinitely splitting Aronszajn tree T with a finitely splitting subtree of uncountable height. Then T is not Lindelöf by Theorem 3.8. $\square$

Recall that Jensen's *diamond principle*, denoted $\diamondsuit$, asserts the existence of a sequence $\langle A_\alpha : \alpha < \omega_1\rangle$ such that $A_\alpha \subseteq \alpha$ and for every $A \subseteq \omega_1$ there exist stationarily many $\alpha < \omega_1$ with $A \cap \alpha = A_\alpha$. We shall need to modify the $\diamondsuit$-sequence so that it "guesses" functions $\omega_1 \to [\omega_1]^{<\omega}$.

Lemma 4.4. *The principle $\diamondsuit$ holds if and only if there is a sequence $\langle f_\alpha : \alpha < \omega_1\rangle$ such that $f_\alpha : \alpha \to [\alpha]^{<\omega}$ and for every $f : \omega_1 \to [\omega_1]^{<\omega}$ the set $\{\alpha : f\restriction\alpha = f_\alpha\}$ is stationary.*

The proof is a standard coding argument, and we omit it.

A sequence $\langle f_\alpha : \alpha < \omega_1\rangle$ of the kind appearing in the statement of Lemma 4.4 will also be referred to as a $\diamondsuit$-sequence.

Recall that an $\aleph_1$-tree is *special* if and only if it can be written as a countable union of antichains. Equivalently, T is special if and only if there is an order preserving map $T \to \mathbb{Q}$, see [6, Lemma III.5.17].

Theorem 4.5. *If $\diamondsuit$ holds, then there is a special Lindelöf tree.*

Proof. We construct a *normal*, infinitely splitting T level by level, together with a specializing function $\varphi : T \to \mathbb{Q}$. To make sure that φ can be extended at limit stages, we require that

$$(*) \qquad \forall x \in T\restriction\alpha\, \forall q \in \mathbb{Q}(\varphi(x) < q \to \exists y \in T_\alpha(x < y \wedge \varphi(y) = q))$$

holds for every $\alpha < \omega_1$. The construction will also depend on a fixed $\diamondsuit$ sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ such that $f_\alpha : \alpha \to [\alpha]^{<\omega}$ and, for every $f : \omega_1 \to [\omega_1]^{<\omega}$, the set $\{\alpha < \omega_1 : f \restriction \alpha = f_\alpha\}$ is stationary. Such a sequence of functions exists by Lemma 4.4.

The underlying set of the tree will be ω_1, with $T \restriction \alpha = \omega\alpha$ and $T_\alpha = [\omega\alpha, \omega(\alpha+1))$ for $\alpha > \omega$. Let 0 be the root of T. For the successor step, given a node at level α we put $\aleph_0$ many nodes immediately above x and let $\varphi \restriction I(x)$ be a bijection between $I(x)$ and $\mathbb{Q} \cap (\varphi(x), \infty)$. Note that normality and $(*)$ continue to hold.

Suppose now that $\alpha < \omega_1$ is a limit ordinal and we've constructed $T \restriction \alpha$ and $\varphi \restriction T \restriction \alpha$. List the set $\{(x, q) \in T \restriction \alpha \times \mathbb{Q} : \varphi(x) < q\}$ as $\{(x_k, q_k) : k \in \omega\}$. The construction splits into two cases.

Case 1: $\omega\alpha > \alpha$. Fix $k \in \omega$. By normality and $(*)$, we can choose a branch b_k with least element x_k such that the heights of members of b_k converge to α and $\sup(\varphi``b_k) = q_k$. Now put a node above b_k and let φ take the value q_k at this node.

Case 2: $\omega\alpha = \alpha$. Fix $k \in \omega$. Choose a cofinal branch $b_k \subseteq T \restriction \alpha$ satisfying the following properties:

(i) $x_k \in b_k$,
(ii) $\sup(\varphi``b_k) = q_k$,
(iii) the unique point on b_k immediately above x_k does not belong to $f_\alpha(x_k)$.

To achieve this, use that, by our construction at successor stages, φ maps $I(x_k)$ onto $\mathbb{Q} \cap (\varphi(x_k), \infty)$ to find a point $z_k \in I(x_k) \setminus f_\alpha(x_k)$ with $\varphi(z_k) < q_k$. Then construct b_k by repeatedly applying $(*)$ and taking a downwards closure. Finally, put a node above b_k on level α and let φ take the value q_k at this node. This completes the construction of T.

It is obvious that φ is a specializing function for T. To see that T is Lindelöf, consider a basic cover $f \in \prod_{x \in T} [I(x)]^{<\omega}$. Pick $\alpha < \omega_1$ limit such that $\omega \cdot \alpha = \alpha$ and $f \restriction \alpha = f_\alpha$, so that $f \restriction (T \restriction \alpha) = f_\alpha$.

The key observation is that every node at level α is in $\uparrow y \setminus \bigcup_{z \in f_\alpha(y)} \uparrow z$ for some $y \in T \restriction \alpha$. By Lemma 3.4, f_α covers T, hence so does $f \restriction \alpha$, and we're done. □

Corollary 4.6. *If $\diamondsuit$ holds, then there is a Lindelöf tree that is not Suslin.*

Proof. Assume $\diamondsuit$ holds. By Theorem 4.5, there is a special Lindelöf tree. But special trees can never be Suslin. □

So, under $\diamondsuit$, we have strict inclusion of our classes of trees:

$$\{\text{Suslin}\} \subsetneq \{\text{Lindelöf}\} \subsetneq \{\text{Aronszajn}\}.$$

5. Adding subtrees

In this section, we address the following question: given an infinitely splitting Aronszajn tree T, can we find a poset $\mathbb{P}$ such that $\Vdash_{\mathbb{P}} T$ is a non-Lindelöf Aronszajn tree? In other words, $\mathbb{P}$ forces the existence of an uncountable finitely splitting subtree of T but at the same time adds no uncountable branches to T.

Definition 5.1. Given an infinitely splitting $\aleph_1$-tree T, we let $\mathbb{P}_T$ be the following poset: conditions are finite functions with $p \in \prod_{x \in F} [I(x)]^{<\omega}$, where $F \in [T]^{<\omega}$, such that $\emptyset \notin \operatorname{ran}(p)$ and which satisfy the following property:

$$(\dagger) \qquad \forall x, y \in \operatorname{dom}(p)(x < y \to y \restriction (\operatorname{ht}(x) + 1) \in p(x)).$$

The order on $\mathbb{P}_T$ is $p \leq q \iff p \supseteq q$. ⌟

The idea is that a condition is a promise that a certain subtree will be finitely splitting at each point of the condition's domain.

Lemma 5.2 (Baumgartner). *Let T be a tree with no uncountable branches. Suppose that $\mathscr{A}$ is an uncountable collection of pairwise disjoint non-empty finite subsets of T. Then there exist $a, b \in \mathscr{A}$ such that for all $x \in a$ and $y \in b$, $x \perp y$.*

For a proof, see [6, Lemma III.5.18].

Lemma 5.3. *Let T be an Aronszajn tree. Then $\mathbb{P}_T$ has the ccc.*

Proof. Let $\langle p_\alpha : \alpha < \omega_1 \rangle \subseteq \mathbb{P}_T$. Letting $d_\alpha = \operatorname{dom}(p_\alpha)$ and thinning out if necessary, we may assume without loss of generality that $\{d_\alpha : \alpha < \omega_1\}$ forms a Δ-system with root R and that $\alpha < \beta < \omega_1$ implies $p_\alpha \restriction R = p_\beta \restriction R$. Apply Lemma 5.2 to find $\alpha < \beta < \omega_1$ such that $x \perp y$ for all $x \in d_\alpha \setminus R$ and $y \in d_\beta \setminus R$. Let $r = p_\alpha \cup p_\beta$. Then r is a condition, because the only way two points in its domain are comparable is if they both belong to either d_α or d_β, hence (†) holds. Since $r \leq p_\alpha, p_\beta$, we are done. □

Lemma 5.4. *Let T be an infinitely splitting Aronszajn tree and $\dot{S}$ a $\mathbb{P}_T$ name for the set $\bigcup_{p \in \dot{G}} \operatorname{dom}(p)$, where $\dot{G}$ is a $\mathbb{P}_T$-name for the generic filter.*

(i) *For every $p \in \mathbb{P}_T$, every $y \in \operatorname{dom}(p)$ and every $x \in T$, if $x < y$, then $p \Vdash x \in \dot{S}$.*
(ii) *$\Vdash \dot{S}$ is downwards closed.*
(iii) *For all $p \in \mathbb{P}_T$ and all $x \in \operatorname{dom}(p)$, $p \Vdash I_{\dot{S}}(x) \subseteq p(x)$.*
(iv) *For all $p \in \mathbb{P}_T$ and all $x \in \operatorname{dom}(p)$, $p \Vdash p(x) \subseteq \dot{S}$.*
(v) *For all $\alpha < \omega_1$, the set $\{p \in \mathbb{P}_T : \exists x \in \operatorname{dom}(p)(\operatorname{ht}(x) \geq \alpha)\}$ is dense in $\mathbb{P}_T$.*

Proof. Write $\mathbb{P} = \mathbb{P}_T$ to simplify the notation.

(i) It suffices to show that the set $\{r \in \mathbb{P} : x \in \operatorname{dom}(r)\}$ is dense below p. Fix $q \leq p$ with $x \notin \operatorname{dom}(q)$. Let $E = \{t \in \operatorname{dom}(q) : x < t\}$. Since $y \in E$, $E \neq \emptyset$. Define a function r with domain $\operatorname{dom}(q) \cup \{x\}$ by $r \restriction \operatorname{dom}(q) = q$ and $r(x) = \{t \restriction (\operatorname{ht}(x)+1) : t \in E\}$. Since $E \neq \emptyset$, $r(x) \neq \emptyset$. Obviously, $q \subseteq r$. We check that r is a condition. Fix $u, v \in \operatorname{dom}(r)$ with $u < v$.
   - If $u, v \in \operatorname{dom}(q)$, then $v \restriction (\operatorname{ht}(u)+1) \in q(u) = r(u)$.
   - If $u = x$, then $v \in E$, so $v \restriction (\operatorname{ht}(u)+1) \in r(x)$ by definition of $r(x)$.
   - If $v = x$, then $u < y$ by $x < y$, therefore

$$\begin{aligned} v \restriction (\operatorname{ht}(u)+1) &= x \restriction (\operatorname{ht}(u)+1) \\ &= y \restriction (\operatorname{ht}(u+1)) \in q(u) = r(u). \end{aligned}$$

   So, in every case, $v \restriction (\operatorname{ht}(u)+1) \in r(u)$, and therefore r is a condition.

(ii) Let G is $\mathbb{P}$-generic over V and $y \in S = \dot{S}_G$. Suppose $x < y$. Fix $p \in G$ with $y \in \operatorname{dom}(p)$. By (i), we can find $q \in G$ with $x \in \operatorname{dom}(p)$. Then $x \in S$.

(iii) Let $G \ni p$ be $\mathbb{P}$-generic over V and suppose $y \in I_S(x)$, where $S = \dot{S}_G$. Since $y \in S$, there exists $q \in G$ with $y \in \operatorname{dom}(q)$. Choose $r \in G$ with $r \leq p, q$. Then $x, y \in \operatorname{dom}(r)$ and $x < y$, so

$$y = y \restriction (\operatorname{ht}(x)+1) \in r(x) = p(x)$$

   because r is a condition.

(iv) Fix $y \in p(x)$. As in (i), it suffices to argue that $\{r \in \mathbb{P} : y \in \mathrm{dom}(r)\}$ is dense below p. Let $q \leq p$ with $y \notin \mathrm{dom}(q)$ and let $E = \{t \in \mathrm{dom}(q) : y < t\}$.

<u>Case 1:</u> $E \neq \emptyset$. This is as in (i): define r by $r\restriction\mathrm{dom}(q) = q$ and $r(y) = \{t\restriction(\mathrm{ht}(y)+1) : t \in E\}$. The verification that r is a condition is exactly the same as in the proof of (i).

<u>Case 2:</u> $E = \emptyset$. Let a be any finite non-empty subset of $I_T(y)$ and let $r = q \cup \{(y, a)\}$. It is enough to show that r is a condition. Fix $u, v \in \mathrm{dom}(r)$ with $u < v$.

- If $u, v \in \mathrm{dom}(q)$, then $v\restriction(\mathrm{ht}(u)+1) \in q(u) = r(u)$.
- If $u = y$, then $v \in \mathrm{dom}(q)$, so $v \in E$, which contradicts $E = \emptyset$.
- If $v = y$, then $u \leq x$ because $y \in I_T(x)$. We now distinguish two cases. <u>If $u = x$</u>, then

$$\begin{aligned} v\restriction(\mathrm{ht}(u)+1) &= y\restriction(\mathrm{ht}(u)+1) \\ &= y \in p(x) = q(x) = r(x) = r(u). \end{aligned}$$

<u>If $u < x$</u>, then $u \in \mathrm{dom}(q)$ and $x \in \mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, so

$$v\restriction(\mathrm{ht}(u)+1) = y\restriction(\mathrm{ht}(u)+1) = x\restriction(\mathrm{ht}(u)+1) \in q(u) = r(u).$$

In every case, $v\restriction(\mathrm{ht}(u)+1) \in r(u)$, and so r is a condition.

(v) Fix $p \in \mathbb{P}$. We may assume that $\mathrm{dom}(p) \subseteq T\restriction\alpha$. Since $\emptyset \notin \mathrm{ran}(p)$, we can define $\gamma = \max\{\mathrm{ht}(y) : y \in \bigcup \mathrm{ran}(p)\}$. Note that $\gamma = \beta + 1$ for some β. Fix $y \in \bigcup \mathrm{ran}(p)$ with $\mathrm{ht}(y) = \beta + 1$. By maximality, $y \notin \mathrm{dom}(p)$.

<u>Case 1:</u> $\beta + 1 = \alpha$. By the proof of (v), there exists $q \leq p$ with $y \in \mathrm{dom}(p)$, and we are done.

<u>Case 2:</u> $\beta + 1 < \alpha$ Since T is normal, there exists $z \in T_\alpha$ with $y < z$. Let a be any finite non-empty subset of $I(z)$ and let $q = p \cup \{(z, a)\}$. We check that q is a condition.

- If $u, v \in \mathrm{dom}(p)$, then it is easy.
- If $u = z$, then $v \in \mathrm{dom}(p)$. But $v > u$, so $\mathrm{ht}(v) > \alpha > \gamma$, contradiction.
- If $v = z$, then $\mathrm{ht}(u) \leq \mathrm{ht}(x)$, where x is the immediate predecessor of y. Since T is a tree, $u \leq x$. If $u = x$, then by $z > y$ we infer that

$$v\restriction(\mathrm{ht}(x)+1) = y \in p(x) = q(x).$$

If $u < x$, then

$$v\restriction(\mathrm{ht}(u)+1) = x\restriction(\mathrm{ht}(u)+1) = p(u) = r(u)$$

In every case, $v\restriction(\mathrm{ht}(u)+1) \in r(u)$, and so r is a condition. □

Theorem 5.5. *Let T be an infinitely splitting Aronszajn tree and $\dot{S}$ a $\mathbb{P}_T$ name for the set $\bigcup_{p \in \dot{G}} \mathrm{dom}(p)$, where $\dot{G}$ is a $\mathbb{P}_T$-name for the generic filter. Then $\Vdash_{\mathbb{P}_T}$ "$\dot{S}$ is an uncountable finitely splitting subtree of T".*

Proof. By Lemma 5.3, $\mathbb{P}_T$ preserves cardinals. By parts (ii),(iii) and (v) of Lemma 5.4, it is forced by $\mathbb{P}_T$ that $\dot{S}$ is an uncountable finitely splitting subtree of T. □

Since the forcing $\mathbb{P}_T$ is only ccc and not obviously Knaster, it is unclear whether T remains Aronszajn in the $\mathbb{P}_T$-extension. To deal with this issue, we first do some preliminary forcing.

Definition 5.6 (Baumgartner)**.** If T is a tree, then $\mathbb{S}_T$ is the poset of finite order preserving partial functions from T into $\mathbb{Q}$ (the set of rational numbers), ordered by inclusion. ⌟

Theorem 5.7 (Baumgartner)**.** *Let T be an Aronszajn tree. Then*

(1) $\mathbb{S}_T$ has the ccc.
(2) $\Vdash_{\mathbb{S}_T} T$ is special.

For a proof, see [6, Lemma III.5.19].

Theorem 5.8. *Let T be an Aronszajn tree. Then $\mathbb{P}_T$ adds no cofinal branches to T. Therefore, $\mathbb{P}_T$ forces that T is a non-Lindelöf Aronszajn tree.*

Proof. Let G be $\mathbb{P}_T$-generic over V. Suppose towards a contradiction that T is not an Aronszajn tree in $V[G]$. Note that the definitions of $\mathbb{S}_T$ and $\mathbb{P}_T$ are both absolute, so the two posets are the same whether computed in V or in any outer model. Let H be $\mathbb{S}_T$-generic over $V[G]$. By the Product Lemma, $V[G][H] = V[H][G]$ and G is $\mathbb{P}_T$-generic over $V[H]$. In V, T has no uncountable branches, so $\mathbb{S}_T$ is ccc, and therefore T is special in $V[H]$. In particular, T is an Aronszajn tree in $V[H]$, hence $\mathbb{P}_T$ is ccc in $V[H]$. But if T is special then it remains special in any $\aleph_1$-preserving extension, in particular T is special in $V[H][G]$. This contradicts the assumption that there is a cofinal branch through T in $V[G]$.

The second assertion in the theorem statement is an immediate consequence of the first and Theorem 5.5. □

Corollary 5.9. *Suppose that $\mathsf{MA}_{\aleph_1}$ holds. Then there are no Lindelöf trees.*

Proof. Given an Aronszajn tree T, we only need to meet $\aleph_1$-many dense sets of $\mathbb{P}_T$ to obtain an uncountable finitely splitting subtree of T, namely those in Lemma 5.4(vi). □

Open questions

(1) Suppose $\lambda < \mu$ are infinite cardinals. Is it consistent, modulo large cardinals, that there exists a cardinal $\kappa \leq \lambda$ which is λ-square compact but not μ-square compact?
(2) For which pairs of infinite cardinals κ, λ with $\kappa < \lambda$ can one find a Hausdorff κ-compact space of weight λ?
(3) Does ZFC prove the existence of a non-Lindelöf special tree?
(4) Is the topological square of a Lindelöf tree Lindelöf?

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