Higher Lindelöf trees

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- A *branch* in a tree is a maximal, totally ordered subset of T. A branch is said to be *cofinal* if it meets every level of T.
- Given a regular, uncountable cardinal κ, a *κ-Aronszajn tree* is a κ-tree without any cofinal branches.

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Warning: A tree can be neither finitely nor infinitely splitting.

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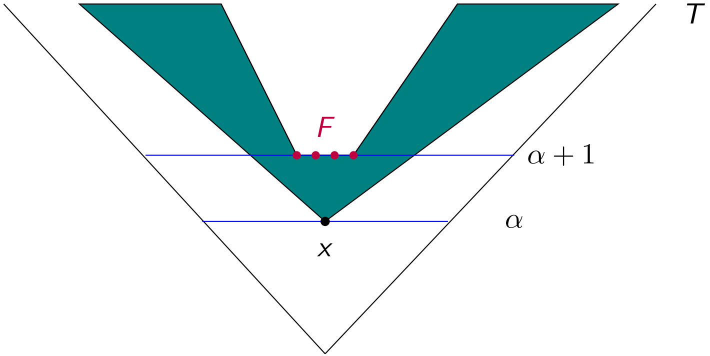

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Question

Let κ be an infinite cardinal and T be a κ-splitting κ^+-tree. When is T κ^+-Lindelöf (with respect to the κ-fine-wedge topology)?

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Theorem (M.)

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Corollary

Every κ-splitting κ^+-Lindelöf tree is κ^+-Aronszajn.

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The situation at $\aleph_1 = \aleph_0^+$

Theorem (M. 2023)

- $\aleph_1$-*Suslin* $\implies$ $\aleph_1$-*Lindelöf* $\implies$ $\aleph_1$-*Aronszajn*

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Theorem (Krueger, 2024)

- *If there is a Suslin tree, then there is a $\aleph_1$-Lindelöf tree that is not Suslin.*

The situation at $\aleph_1 = \aleph_0^+$

Theorem (M. 2023)

- *$\aleph_1$-Suslin $\implies$ $\aleph_1$-Lindelöf $\implies$ $\aleph_1$-Aronszajn*

Recall that a κ^+-tree is special if it can be written as a union of κ-many antichains.

- *There is a special Aronszajn tree that is not $\aleph_1$-Lindelöf.*
- *If $\diamondsuit$ holds, then there is a special $\aleph_1$-Lindelöf tree.*
- *$MA_{\aleph_1}$ implies that there are no $\aleph_1$-Lindelöf trees.*

Theorem (Krueger, 2024)

- *If there is a Suslin tree, then there is a $\aleph_1$-Lindelöf tree that is not Suslin.*
- *It is consistent that CH holds and there are no $\aleph_1$-Lindelöf trees.*

Question

What happens at larger cardinals?

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For any infinite κ, we still have that

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Theorem (M. 2024)

Assume that $\diamondsuit_{\kappa^+}$ and $\kappa^{<\kappa} = \kappa$ hold. Then there is a special, κ^+-Lindelöf tree.

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What happens at successors of singulars?

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The *proxy principles*, developed by Brodsky and Rinot, allow one to construct a wide variety of κ-Suslin trees, via the so-called *microscopic approach*.

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An instance of the proxy principle $\mathrm{P}_\xi(\kappa, \mu, \mathcal{R}, \theta, \mathcal{S}, \nu, \sigma)$ involves 8 (eight) parameters.

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Theorem (Brodsky-M.)

Let κ *be an infinite cardinal. Assume that* $\mathrm{P}_\kappa(\kappa^+, \kappa^+, \sqsubseteq, 1, \{\kappa^+\}, \kappa^+, 1)$ *holds. Then there is a special* κ^+*-Lindelöf tree.*

$\aleph_2$

Question

Is it consistent that there is an $\aleph_2$-Aronszajn tree, but no $\aleph_2$-Lindelöf trees?

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Given an $\aleph_1$-splitting $\aleph_2$-tree T, consider the poset $\mathbb{P}$ of partial countable functions $p : T \rightharpoonup [T]^{\leq\aleph_0} \setminus \{\emptyset\}$ satisfying that

$$\forall x, y \in \mathrm{dom}(p) \left[x < y \to y \restriction (\mathsf{ht}(x) + 1) \in p(x)\right].$$

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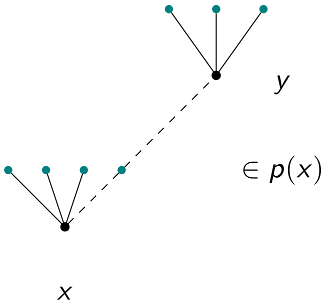

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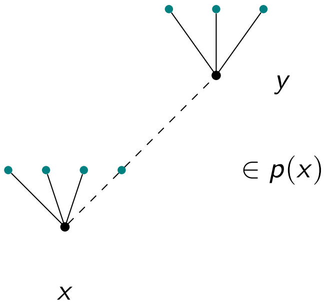

Order $\mathbb{P}$ by reverse inclusion.

$\aleph_2$

Forcing with $\mathbb{P}$ adds a $<\aleph_2$-splitting subtree of height $\aleph_2$ to T.

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Theorem (Brodsky-M. 2026)

Assume there is a weakly compact cardinal and CH holds. Then there is a forcing extension in which there is an $\aleph_2$-Aronszajn tree but no $\aleph_2$-Lindelöf trees.

Thank you :)