Square compactness and dense linear orders

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Preliminaries

All spaces under consideration are Hausdorff.

Definition

1. Given an infinite cardinal κ, a topological space is said to be κ-*compact* iff every open cover has a subcover of size $< \kappa$.
2. A cardinal κ is said to be λ-*square compact* iff for every topological space X of weight $\leq \lambda$, if X is κ-compact, then X^2 is κ-compact.
3. A cardinal κ is said to be *weakly square compact* iff it is κ-square compact.
4. (Hajnal-Juhász, 1973) A cardinal κ is said to be *square compact* iff it is λ-square compact for every cardinal λ.

Theorem (Hajnal-Juhász, 1973)

If κ is weakly square compact (i.e. κ-square compact), then κ is regular.

Theorem (Hajnal-Juhász, 1973)

If κ is uncountable and 2^κ-square compact, then κ is weakly compact.

Theorem (folklore)

If κ is strongly compact, then it is square compact.

Question (Juhasz)

Does the previous theorem reverse?

Theorem (Buhagiar-Džamonja, 2020)

Let κ be an uncountable cardinal such that $\kappa = \kappa^{<\kappa}$. The following are equivalent:

1. *κ is weakly compact.*
2. *κ is weakly square compact.*

The goal of this talk is to show that the assumption $\kappa = \kappa^{<\kappa}$ is superfluous.

The Sorgenfrey construction

Recall the classical argument that Lindelöfness is not productive: endow $\mathbb{R}$ with the Sorgenfrey topology, which is generated by all intervals $[a, b)$, where $a, b \in \mathbb{R}$. This is obviously Hausdorff.

Claim

The Sorgenfrey topology is Lindelöf.

Proof.

If $\mathcal{U} \subset \{[a, b) : a, b \in \mathbb{R}\}$ is a basic open cover, then $W = \bigcup\{(a, b) : [a, b) \in \mathcal{U}\}$ is Euclidean open, hence $W = \bigcup\{(a, b) : [a, b) \in \mathcal{U}_0\}$ for some countable $\mathcal{U}_0 \subset \mathcal{U}$. Argue that $\mathbb{R} \setminus W$ is countable: if $x \in \mathbb{R} \setminus W$, then $x \in [a_x, b_x)$ for some $[a_x, b_x) \in \mathcal{U}$, hence $x = a_x$. Pick $d_x \in \mathbb{Q}$ with $x < d_x < b_x$. If $x, y \in \mathbb{R} \setminus W$, $x < y$, then $x < d_x < b_x \leq y < d_y$. □

In the square of the Sorgenfrey topology, the set $\{(x, -x) : x \in \mathbb{R}\}$ is closed, discrete, and has size $2^{\aleph_0}$, hence $\mathbb{R} \times \mathbb{R}$ is non-Lindelöf.

A generalization

The previous argument can be extended to any infinite cardinal κ (instead of $\aleph_1$), as long as we have a dense linear order (dlo) X with $d(X) < \kappa = |X|$, where $d(X) = \min\{|D| : D \subset X \text{ dense}\}$ is the density of X.

Theorem

Let κ be an uncountable cardinal. If there exists a dlo X with $d(X) < \kappa = |X|$, then κ is not weakly square compact.

Question

Given κ uncountable, is there a dlo X with $d(X) < \kappa = |X|$?

The Three Cardinal Lemma

Theorem (M.)

Let κ be an uncountable cardinal. Suppose there exist cardinals μ and θ such that $\mu^{<\theta} = \mu < \kappa \leq \mu^{\theta}$. Then there is a dlo X with $d(X) < \kappa = |X|$.

Corollary (M.)

Successor cardinals are not weakly square compact.

Proof of Corollary.

Fix $\lambda \geq \omega$. Let $\theta := \min\{\nu : \lambda^{\nu} > \lambda\}$. By König's Lemma, $\theta \leq \mathrm{cf}(\lambda)$, so $\lambda^{<\theta} = \lambda$ by minimality. Apply the theorem with $\kappa = \lambda^{+}$ and $\mu = \lambda$. □

Corollary

Suppose κ is weakly square compact. Then κ is a strong limit. In particular, $\kappa^{<\kappa} = \kappa$.

Proof sketch.

Suppose κ is not a strong limit. Then the following makes sense:

$$\theta := \min\{\nu : \exists\lambda(\nu \leq \lambda < \kappa \leq \lambda^{\nu})\}$$

Let $\lambda < \kappa$ be least such that $\theta \leq \lambda < \kappa \leq \lambda^{\theta}$. Put $\mu := \lambda^{<\theta}$.

First, argue that θ is regular. Then, show that $\mu^{<\theta} = \mu < \kappa$ by splitting into the cases θ successor and θ limit, and then casing on the behaviour of $\alpha \mapsto \lambda^{\alpha}$ for $\alpha < \theta$. $\square$

We can now remove the assumption $\kappa^{<\kappa} = \kappa$ from the Buhagiar-Džamonja result:

Theorem

Let κ be an uncountable cardinal. The following are equivalent:

1. *κ is weakly compact.*
2. *κ is κ-square compact.*

Proof.

Both (1) and (2) separately imply $\kappa^{<\kappa} = \kappa$. □

The Three Cardinal Lemma: proof sketch

Theorem

Let κ be uncountable. Suppose there exist cardinals μ and θ such that $\mu^{<\theta} = \mu < \kappa \leq \mu^{\theta}$. Then there is a dlo X with $d(X) < \kappa = |X|$.

Lemma 1

Given $\mu \geq \omega$ and any dlo $(X, <)$, there exists a dlo $Y \supset X$ such that any non-empty interval in Y with endpoints from X contains μ many pairwise disjoint non-empty intervals. If $|X| \leq \mu$, then we can find Y with $|Y| = \mu$.

Proof of Lemma 1.

Let $X = \{x_\alpha : \alpha < \delta\}$, for $\delta = |X|$. Define $\langle X_\alpha : \alpha \leq \delta \rangle$ as follows: $X_0 := X$ and $X_\alpha = \bigcup_{\xi<\alpha} X_\xi$ for limit α. Given X_α, let $S = (\mu^* \oplus \mu) \times \mathbb{Q}$ ordered lexicographically and put

$$X_{\alpha+1} = \{x \in X_\alpha : x \leq x_\alpha\} \oplus S \oplus \{x \in X_\alpha : x_\alpha < x\}.$$

□

Interlude

Definition

Let $(P, <)$ be a strict partial order. Given cardinals κ, λ, a (κ, λ)-*pregap* in $(P, <)$ is a pair (f, g) such that $f : \kappa \to P$ is strictly increasing, $g : \lambda \to P$ is strictly decreasing, and $f(\alpha) < g(\beta)$ for all $\alpha < \kappa$ and $\beta < \lambda$. We say that $p \in P$ *fills* the pregap if $f(\alpha) < p < g(\beta)$ for all $\alpha < \kappa$ and $\beta < \lambda$.

A *gap* is a pregap which is not filled by any element of P.

Lemma 2

Let $(X, <)$ be a dlo and (f, g) a gap in X. Then there is a dlo $Y \supset X$ with $|Y| = |X|$ in which (f, g) is filled.

Proof of Lemma 2.

Use the Compactness Theorem. □

Lemma 3

Suppose μ and θ are infinite cardinals such that $\mu^{<\theta} = \mu$. Let $(X, <)$ be a dlo with $|X| = \mu$. Then there is some dlo $Z \supset X$ such that $|Z| = \mu$ and every $(<\theta, <\theta)$-pregap in Z is filled.

Proof of lemma 3.

Enumerate the $(<\theta, <\theta)$-gaps (there's μ-many) and iterate Lemma 2. □

Proof of the Three Cardinal Lemma.

1. Build a $\subset$-increasing sequence $\langle X_\alpha : \alpha \leq \mu \rangle$ of dlo's of size μ as follows: X_0 has every interval contain μ-many pairwise disjoint intervals. At limits, take unions. At successors, use Lemma 1 to enlarge X_α, then apply Lemma 3 to fill all relevant gaps of the enlargement, call the output $X_{\alpha+1}$.
2. In X_μ, find a coherent system of non-empty closed intervals $\langle I_s : s \in {}^{<\theta}\mu \rangle$, i.e. $s \subset t \to I_s \supset I_t$ and $s \perp t \to I_s \cap I_t = \emptyset$. If Y is the order completion of X_μ, argue that $|Y| \geq \mu^\theta > \kappa$.
3. Choose $X \prec Y$ with $X_\mu \subset X$ and $|X| = \kappa$. Since X_μ is dense in Y, $d(X) \leq |X_\mu| = \mu < \kappa$. $\square$

Thank you :)

References

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