An inadequate course in cardinal functions

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1 Definition. Let (X, τ) be a topological space. We define the following global cardinal functions:

- ☞ The **weight** of X, $w(X) := \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base for } X\}$.
- ☞ The **density** of X, $d(X) := \min\{|D| : D \subseteq X \text{ is dense }\}$.
- ☞ The **spread** of X, $s(X) := \sup\{|S| : S \subseteq X \text{ is discrete}\}$.
- ☞ The **extent** of X, $e(X) := \sup\{|C| : C \subseteq X \text{ is closed and discrete}\}$.
- ☞ The **Lindelöf degree** of X,

$$L(X) := \min\{\kappa : \text{every open cover of } X \text{ has a subcover of size } \leq \kappa\}$$

- ☞ The **network weight** of X, $nw(X) = \min\{|\mathcal{N}| : \mathcal{N} \text{ is a network for } X\}$, where a **network** is a set $\mathcal{N} \subseteq \mathcal{P}(X)$ such that whenever $x \in U \in \tau$, there is some $N \in \mathcal{N}$ with $x \in N \subseteq U$.

We also have the following local functions, defined for $p \in X$,

- ☞ $\psi(X, p) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a local base at } p\}$, where $\mathcal{V}$ is a **local base** at p iff $\mathcal{V} \subseteq \tau$ and whenever $p \in U \in \tau$, then $p \in V \subseteq U$ for some $V \in \mathcal{V}$. In other words, $\mathcal{V}$ is a base of open neighbourhoods of p.
- ☞ $\psi(X, p) = \min\{|\mathcal{V}| : \mathcal{V} \text{ is a pseudo base at } p\}$, where $\mathcal{V}$ is a **pseudo base** at p iff $\mathcal{V} \subseteq \tau$ and $\bigcap \mathcal{V} = \{p\}$.
- ☞ $t(X, p) = \min\{\kappa : \forall Y \subseteq X(x \in \overline{Y} \to \exists A \in [Y]^{\leq\kappa}(x \in \overline{A}))\}$.

These in turn give rise to global functions:

- ☞ The **character** of X, $\chi(X) = \sup\{\chi(X, p) : p \in X\}$.
- ☞ The **pseudocharacter** of X, $\psi(X) = \sup\{\psi(X, p) : p \in X\}$.
- ☞ The **tightness** of X, $t(X) = \sup\{t(X, p) : p \in X\}$. ⌟

An easy result:

2 Lemma. *Let X be a space. Then $e(X) \leq L(X) \leq w(X)$. If X is T_0, then $|X| \leq 2^{w(X)}$.*

Proof. Fix $E \subseteq X$ closed and discrete with $|E| = e(X)$. Then $\{\{x\} : x \in E\} \cup \{X \setminus E\}$ is an open cover of X with no subcover of size $< e(X)$, hence $e(X) \leq L(X)$. For the second inequality, fix a base $\mathcal{B}$ with $|\mathcal{B}| = w(X)$. If $\mathcal{U}$ is an open cover of X, now pick $\mathcal{V} \subseteq \mathcal{B}$ such that every member of $\mathcal{V}$ is a subset of a member of $\mathcal{U}$ (i.e. $\mathcal{V}$ refines $\mathcal{U}$) and $\bigcup \mathcal{V} = X$.

Finally, if X is T_0 and $\mathcal{B}$ is as above, $x \mapsto \langle \chi_U(x) : U \in \mathcal{B} \rangle$ is an injection $X \to {}^{\mathcal{B}}2$. ■

3 Lemma. *Consider the discrete topology on an infinite cardinal κ. Put $Y = {}^{\kappa}2$. Then $d(\kappa^Y) = \kappa$ (in the product topology).*

Proof. Consider the product topology on Y, so that $w(Y) = \kappa$. Let $\mathcal{B}$ be the usual basis for Y. Let D be the set of $p \in \kappa^Y$ defined as follows: $p \in D$ iff there is some finite non-empty $\mathscr{A} \subseteq \mathcal{B}$ consisting of pairwise disjoint sets such that p is constant on each member of $\mathscr{A}$ and $p[Y \setminus \bigcup \mathscr{A}] = \{0\}$.

Claim. *D is dense in κ^Y.*

Proof of claim. Fix a basic open set $U \subseteq \kappa^Y$, so that there exist $F \in [Y]^{<\omega}$ and sets $V_y \subseteq \kappa$ for $y \in F$ so that $U = \prod_{y \in Y} U_j$, where $U_y = V_y$ if $y \in F$ and $U_y = \kappa$ if $y \in Y \setminus F$.

Since Y is Hausdorff, we can find, for each $y \in F$, $B_y \in \mathcal{B}$ such that $y \in B_y$ and the $B_y \cap B_z = \emptyset$ whenever $y \neq z$. Choose, for each $y \in F$, an ordinal $\alpha_y \in V_y$. Define $p \in \kappa^Y$ by

$$p(y) := \begin{cases} \alpha_y & \text{if } y \in B_y, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, $p \in D$. Also, if $y \in F$, then $p(y) = \alpha_y \in V_y$, so $p \in U$. $\dashv$

To complete the proof, simply note that $|D| = \kappa$. ■

4 Theorem (Hewitt-Marczewski-Pondiczery)**.** *Let κ be infinite, J a set of size $\leq 2^{\kappa}$, and X_j spaces with $d(X_j) \leq \kappa$ for all $j \in J$. Then $d(\prod_j X_j) \leq \kappa$. In particular, the product of continuum many separable spaces is separable.*

Proof. Fix, for each $j \in J$, $D_j \subseteq X_j$ dense with $|D_j| = d(X_j) \leq \kappa$. Choose surjections $f_j : \kappa \to D_j$, which induce a continuous map $f = \prod_j f_j : \kappa^Y \to \prod_j D_j$. Since f is surjective and continuous, the previous lemma gives $d(\prod_j D_j) \leq \kappa$. But $\prod_j D_j$ is obviously dense in $\prod_j X_j$, and we are done. ■

5 Jones' Lemma. Let X be a T_4 space and E a closed discrete subset of X. Then $2^{d(X)} \leq 2^{|E|}$.

Proof. Fix E closed and discerete and $D \subseteq X$ dense with $|D| = d(X)$. Given $A \subseteq E$, the characteristic function of A as computed in E is continuous (because E is discrete), hence extends to some $f_A \in C(X, [0,1])$ by the Tietze extension theorem (using that E is closed). Note that, if $A \neq B$, then $f_A \neq f_B$, hence $f_A \restriction D \neq f_B \restriction D$. This gives an injection $\mathcal{P}(E) \to {}^{D}[0,1]$. The codomain has size $2^{d(X)}$, which completes the proof. ■

By the same argument at the end of the last proof:

6 Lemma. *Let X be a space. Then $|C(X, \mathbb{R})| \leq 2^{d(X)}$.*

The seemingly strange concept of network weight is useful for computing the weight of compact spaces, via the following surprising result:

7 Theorem (Arhangel'skii)**.** *Let (X, τ) be a compact Hausdorff space. Then $nw(X) = w(X)$.*

Proof. Every base is a network, so $w(X) \geq nw(X)$. For the reverse implication, fix a network $\mathcal{N}$ with $|\mathcal{N}| = nw(X)$. Define a partial function $\Phi : \mathcal{N}^2 \rightharpoonup \tau^2$ as follows: given $(S,T) \in \mathcal{N}^2$ with $S \cap T = \emptyset$, if there exist open sets U and V with $U \supseteq S$, $V \supseteq T$, and $U \cap V = \emptyset$, then $\Phi(S,T)$ chooses one such pair (U,V). Let $\mathscr{A} = \mathrm{dom}(\mathrm{ran}(\Phi))$, so that $\mathscr{A}$ is the collection of all open sets which occur as the first (or second, by symmetry) coordinate of a value of Φ. Let $\mathcal{B}$ be the set of finite intersections of members of $\mathscr{A}$.

Claim. *$\mathcal{B}$ is a basis for X.*

Proof of claim. Fix $x \in U \in \tau$. Given $y \in X \setminus U$, choose $U_y, V_y \in \tau$ with $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \emptyset$. Since $\mathcal{N}$ is a network, we can choose $S_y, T_y \in \mathcal{N}$ such that $x \in S_y \subseteq U_y$ and $y \in T_y \subseteq V_y$. This shows that Φ is defined at (S_y, T_y), say $\Phi(S_y, T_y) = (U_y^1, V_y^1)$, so that $U_y^1, V_y^1 \in \mathscr{A}$.

Since X is compact and U is open, $X \setminus U$ is compact. Also, $X \setminus U \subseteq \bigcup_{y \in X \setminus U} V_y^1$, so we can find $y_0, \ldots, y_{n-1}$ such that $X \setminus U \subseteq \bigcup_{i<n} V_{y_i}^1$. Since $U_y^1 \cap V_y^1 = \emptyset$ by definition of Φ, we have that

$$x \in \bigcap_{i<n} U_{y_i}^1 \subseteq \bigcap_{i<n} X \setminus V_{y_i}^1 \subseteq U.$$

As $\bigcap_{i<n} U_{y_i}^1 \in \mathcal{B}$, we are done. ⊣

Obviously, $|\mathcal{B}| = |\mathscr{A}| \leq |\mathcal{N}| = nw(X)$, so $w(X) \leq nw(X)$. ■

8 Weight Addition Theorem (Arhangel'skii)**.** Let be (X,τ) a compact Hausdorff space. Suppose that $X = \bigcup_{\alpha<\kappa} X_\alpha$, where $w(X_\alpha) \leq \kappa$ for every $\alpha < \kappa$. Then $w(X) \leq \kappa$.

Proof. Fix, for each $\alpha < \kappa$, a base $\mathcal{B}_\alpha$ for X_α with $|\mathcal{B}_\alpha| \leq \kappa$. Put $\mathcal{N} = \bigcup_{\alpha<\kappa} \mathcal{B}_\alpha$. Suppose $x \in U \in \tau$. Fix $\alpha < \kappa$ with $x \in X_\alpha$. Choose $B \in \mathcal{B}_\alpha$ such that $x \in B \subseteq X_\alpha \cap U$. Obviously, $x \in B \subseteq U$. This shows that $\mathcal{N}$ is a network for X, and so $w(X) = nw(X) \leq |\mathcal{N}| \leq \kappa$. ■

9 Remark. To see that compactness is necessary, let X be a countable T_1 space with uncountable weight. Examples of such monstrosities include the Appert space and the Arens-Fort space, see [8], pages 117 and 54 respectively. Now write X as a countable union of finite sets and use that a finite T_1 space has finite weight. ⌟

The following corollary is the first step towards some results about Stone-Čech compactifications.

10 Corollary. *Let X and Y be spaces, with Y compact Hausdorff. Suppose that there exists a continuous surjection $f : X \to Y$. Then $w(Y) \leq w(X)$.*

Proof. Let $\mathcal{B}$ be a base for X with $w(X) = |\mathcal{B}|$. Then $\{f[B] : B \in \mathcal{B}\}$ is a network for Y of size at most $w(X)$. ■

11 Lemma. *If X is a $T_{3\frac{1}{2}}$ space, then $w(\beta X) \leq 2^{d(X)}$.*

Proof. Construe βX as a subspace of $[0,1]^{C(X,[0,1])}$ (see [3] for details). This latter space can be viewed as a subspace of $[0,1]^{2^{d(X)}}$ (by lemma 6), and this has weight $2^{d(X)}$. ■

12 Corollary. $w(\beta\omega) = 2^{\aleph_0}$.

Proof. The ($\leq$) inequality follows from the previous lemma. To see the reverse inequality, first note that $nw(\beta\omega) = w(\beta\omega)$ by Arhangel'skii's theorem, so it suffices to show $nw(\beta\omega) \geq 2^{\aleph_0}$.

By the Hewitt-Marczweski-Pondiczery theorem, $I^{I^\omega} \cong I^I$ is separable. Fix $D \subseteq I^I$ dense and countable and a bijection $f : \omega \to D$. Then $f : \omega \to I^I$ is continuous, so it extends to some continuous $\beta f : \beta\omega \to I^I$ by the compactness of I^I and general abstract nonsense. By continuity, the range of βf is compact, hence closed (because $\beta\omega$ is Hausdorff). But $D \subseteq \mathrm{ran}(\beta f)$, so $\mathrm{ran}(\beta f) = \beta\omega$ and so $nw(\beta\omega) \leq nw(I^I) = w(I^I) = 2^{\aleph_0}$. ■

We remark that, since $\beta f : \beta\omega \to I^I$ is surjective, the proof also gives that $|\beta\omega| \geq 2^{2^{\aleph_0}}$. This argument is independent of the ultrafilter argument using independent sets.

The next theorem answers a question of Alexandroff and Urysohn from 1923: if a space is Hausdorff, first countable, and compact, must it be of size at most continuum?

13 Theorem (Arhangel'skii, 1969)**.** *Let X be a Hausdorff space with $L(X) = t(X) = \psi(X) = \aleph_0$. Then $|X| \leq \aleph_0$.*

In fact, Arhangelsk'ii proved a stronger result, namely that $|X| \leq 2^{L(X)\chi(X)}$ whenever X is T_1, but we will prove the version above to ilustrate the use of model theoretic methods. The interested reader can consult page 19 of [4].

Our proof follows [1]. While elementary submodels abstract Pol's closure proof, I do not know who first used them in this context.

Proof. Fix $\mathcal{B}$ a base for X. Let λ be a large enough regular cardinal so that $(X, \mathcal{B}) \in H_\lambda$. Choose $M \prec H_\lambda$ so that ${}^\omega M \subseteq M$, $(X, \mathcal{B}) \in M$, and $|M| = 2^{\aleph_0}$. The strategy of the proof is to show that $X \cap M = X$, which will in particular establish the desired cardinality bound. Suppose therefore that $X \cap M \neq X$,and fix $z \in X \setminus M$.

Claim 1. *For every $y \in X \cap M$ there is some $U_y \in \mathcal{B} \cap M$ such that $y \in U_y$ and $z \notin U_y$.*

Proof of claim 1. Fix $y \in X \cap M$. Since X has countable pseudcharacter, the elementarity of M gives $\mathcal{U} \in M \cap [\mathcal{B}]^\omega$ such that $\bigcap \mathcal{U} = \{y\}$. Choose $U_y \in \mathcal{U}$ such that $z \notin U_y$. Since $\mathcal{U} \in M$ is countable, $\mathcal{U} \subseteq M$ by elementarity, so $U_y \in M$. ⊣

The next step is to show that $X \cap M$ is closed in X. Fix $x \in \overline{X \cap M}$. Since X has countable tightness, there is some countable $Y \subseteq X \cap M$ such that $x \in \overline{Y}$. Use the countable pseudocharacter to find $\langle V_n : n < \omega \rangle \in {}^\omega\mathcal{B}$ such that $\{x\} = \bigcap_n V_n$. Now fix $n \in \omega$.

Claim 2. *There is a family $\langle V_{n,m} : n, m \in \omega \rangle$ such that $X \setminus V_n \subseteq \bigcup_{m\in\omega} V_{n,m}$ and $x \notin \bigcup_{m\in\omega} \overline{V_{n,m}}$.*

Proof of claim 2. Fix $y \in X \setminus V_n$, so that $x \neq y$ and we can find $V_y, W_y \in \mathcal{B}$ such that $x \in W_y, y \in V_y$, and $V_y \cap W_y = \emptyset$. Since V_n is open and X is Lindelöf, $X \setminus V_n$ is Lindelöf, so we can find $\{y_{n,m} : m \in \omega\} \subseteq X \setminus V_n$ such that $X \setminus V_n \subseteq \bigcup_{m\in\omega} V_{y_{n,m}}$. Put $V_{n,m} := V_{y_{n,m}}$. This works. ⊣

Let $Y_{n,m} := Y \cap V_{n,m}$.

Claim 3. $\overline{Y} \setminus \{x\} = \bigcup_{n,m\in\omega} \overline{Y_{n,m}}$.

Proof of claim 3. $\subseteq$) Let $y \in \overline{Y} \setminus \{x\}$. Choose $n \in \omega$ such that $y \in X \setminus V_n$, and then choose $m \in \omega$ with $y \in V_{n,m}$. Since $V_{n,m}$ is open and $y \in \overline{Y}$, we infer that $y \in \overline{Y \cap V_{n,m}} = \overline{Y_{n,m}}$. This is a general topology fact: if $A \subseteq X$ and V is open, then $V \cap \overline{A} \subseteq \overline{V \cap A}$. Indeed, if $p \in V \cap \overline{A}$ and W is an open neighbourhood of p, then so is $V \cap W$, so $A \cap (V \cap W) \neq \emptyset$ by $p \in \overline{A}$.

$\supseteq$) Fix $n, m \in \omega$ and $y \in \overline{Y_{n,m}}$. Since $x \notin \overline{V_{n,m}}$, we have $y \neq x$. Also, $\overline{Y_{n,m}} \subseteq \overline{Y}$, so $y \in Y \setminus \{x\}$. $\dashv$

Now, $Y \subseteq X \cap M$ is countable, so $Y \in M$ because M is closed under countable sequences, hence $\overline{Y} \in M$ by elementarity. By the same logic, $Y_{n,m} \in M$ and $\overline{Y_{n,m}} \in M$ for every $n, m \in \omega$. Using closure under ω-sequences once more, we see that $\langle \overline{Y_{n,m}} : n, m \in \omega \rangle \in M$. By elementarity and claim 3, $\overline{Y} \setminus \{x\} \in M$. But $x \in \overline{Y} \in M$, so $x \in M$ by elementarity once more. Since $x \in \overline{X \cap M}$ was arbitrary, $X \cap M$ is closed.

Since X is Lindelöf and $X \cap M$ is closed, we infer that $X \cap M$ is also Lindelöf (in the subspace topology). Put $\mathcal{U} = \{U_y : y \in X \cap M\}$, where the U_y are as in claim 1. Then $\mathcal{U}$ is an open cover of $X \cap M$, hence admits an countable subcover $\mathcal{U}_0$. Since $\mathcal{U} \subseteq M$ and M is closed under ω-sequences, we have $\mathcal{U}_0 \in M$. The fact that $\mathcal{U}_0 = \mathcal{U}_0 \cap M$ covers $X \cap M$ says that the statement

$$\forall y \in X \cap M \exists U \in \mathcal{U}_0 \cap M (y \in U)$$

is true. But this means exactly that $M \models (\mathcal{U}_0 \text{ covers } X)$, so $\mathcal{U}_0$ covers X by elementarity (and absoluteness). This is a contradiction, because $z \notin \bigcup \mathcal{U}_0$ (by claim 1) yet $z \in X$. ■

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