Kaplansky's Conjecture

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Please let me know if you spot any errors!

Contents

1 Preliminaries and outline

Let KC be the statement "For every compact Hausdorff space X and every Banach algebra A there is no discontinuous algebra homomorphism $C(X, \mathbb{C}) \to A$.". Here, $C(X, \mathbb{C})$ denotes the set of complex valued continuous functions $X \to A$. One can also consider real valued functions (in which case A must be a real algebra), both statements are equivalent.

The question of whether KC was originally raise (in a slightly different form) by Kaplansky, and became known as "Kaplansky's Conjecture" (there is also an algebraic problem of the same name).

In 1976, Dales and Esterle independently proved that CH implies $\neg$KC (for a proof, see Theorem 5.7.20 on page 771 of [1]) .

The aim of these notes is to present (part of) a proof of the following result:

1.1 Theorem (Solovay-Woodin)**.** *If* ZFC *is consistent, then so is the* $\mathrm{ZFC} + \mathrm{MA}(\aleph_1) + \mathrm{KC}$.

Our treatment mostly follows [4], although we fix some issues (hopefully without introducing new ones) and provide more (too many?) details. An alternative treatment can be found in [2], where all forcing arguments are stated in the language of Boolean algebras.

To state the key theorem which takes us from the world of Banach algebras into set theory, we need some terminology.

1.2 Definition. Let $f, g \in \omega^\omega$.

☞ $f < g \iff \forall n \in \omega(f(n) < g(n))$.

☞ $f \leq g \iff \forall n \in \omega(f(n) \leq g(n))$. Note that $<$ is *not* the strict part of $\leq$.

☞ $f <_{\mathcal{F}} g \iff \{n \in \omega : f(n) < g(n)\} \in \mathcal{F}$, where $\mathcal{F}$ is a filter on ω. When $\mathcal{F}$ is the Frechet filter (i.e. the filter of cofinite sets), then we write $<^*$ instead of $<_{\mathcal{F}}$. Note that

$$f <^* g \iff \forall^\infty n \in \omega(f(n) < g(n)) \iff \exists m \in \omega \forall n \geq m(f(n) < g(n))$$

☞ $f \leq_{\mathcal{F}} g \iff \{n \in \omega : f(n) \leq g(n)\} \in \mathcal{F}$, where $\mathcal{F}$ is a filter on ω. Again, we write $\leq^*$ when $\mathcal{F}$ is the Frechet filter. Note that $<_{\mathcal{F}}$ is *not* the strict part of the preorder $\leq_{\mathcal{F}}$.

☞ $f \ll g \iff \lim_{n\to\infty} g(n) - f(n) = \infty$.

We also write $f =_{\mathcal{F}} g \iff \{n \in \omega : f(n) = g(n)\} \in \mathcal{F}$. Again, when $\mathcal{F}$ is the Frechet filter, we write $=^*$ instead of $=_{\mathcal{F}}$. Clearly, $=_{\mathcal{F}}$ is an equivalence relation of ω^ω. We let $[f]_{\mathcal{F}}$ denote the equivalence class of $\mathcal{F}$.

Obviously, if $\mathcal{F}$ and $\mathcal{G}$ are filters, with $\mathcal{F} \subseteq \mathcal{G}$, then $f <_{\mathcal{F}} g$ implies $f <_{\mathcal{G}} g$. In particular, $f <^* g$ implies $f <_{\mathcal{U}} g$ whenever $\mathcal{U}$ is a non-principal ultrafilter. ⌟

Let $\mathcal{U}$ be a non-principal ultrafilter on ω. By elementary model theory, the ultrapower $(\omega^\omega/\mathcal{U}, <_{\mathcal{U}})$ is totally ordered. Given $f \in \omega^\omega$, consider the initial segment

$$S_{\mathcal{U}}(f) := \{[h]_{\mathcal{U}} : h <_{\mathcal{U}} f\}. \tag{1}$$

1.3 Definition. Let $(\mathbb{P}, <_{\mathbb{P}})$ and $(\mathbb{Q}, <_{\mathbb{Q}})$ be strict partial orders. A map $f : \mathbb{P} \to \mathbb{Q}$ is **order preserving** iff for all $p, p' \in \mathbb{P}$,

$$p <_{\mathbb{P}} p' \implies f(p) <_{\mathbb{Q}} f(p')$$

We say f is an **order embedding** iff for all $p, p' \in \mathbb{P}$,

$$p <_{\mathbb{P}} p' \iff f(p) <_{\mathbb{Q}} f(p')$$

If there is such an f, we say that $\mathbb{P}$ embeds into $\mathbb{Q}$.

Note that, if $\mathbb{P}$ is a total order, then every order preserving function is an order embedding. ⌟

1.4 Theorem (Woodin)**.** *Suppose there exist a compact Hausdorff space X, a Banach algebra A and a discontinuous algebra homomorphism $\theta : C(X, \mathbb{C}) \to A$. Then there exist a non-principal ultrafilter $\mathcal{U}$ on ω, a monotone*[1] *and unbounded $g \in \omega^\omega$, and an order preserving function $\pi : (S_{\mathcal{U}}(g), <_{\mathcal{U}}) \to (\omega^\omega, <)$.*

The conclusion of 1.4 is sometimes called "Woodin's Condition". We shall not prove 1.4, as the argument is rather long. We refer the interested reader to [2] for the proof and to [1] for background material in Banach algebras.

So, the strategy to prove 1.1 will be to obtain a model where there is no order preserving function $\pi : (S_{\mathcal{U}}(g), <_{\mathcal{U}}) \to (\omega^\omega, <)$ for any g and $\mathcal{U}$ as above. This is done by considering an intermediate ordering through which our map would factor.

[1] i.e. $n \leq m \to g(n) \leq g(m)$.

Consider the set 2^{ω_1} ordered lexicographically, that is

$$x <_\ell y \iff \exists \alpha < \omega_1 [x \restriction \alpha = y \restriction \alpha \wedge x(\alpha) = 0 \wedge y(\alpha) = 1]$$

It is plain that $(2^{\omega_1}, <_\ell)$ is totally ordered.

1.5 Theorem. *Assume* $\mathrm{MA}(\aleph_1)$. *Let* $\mathcal{U}$ *be a non-principal ultrafilter on* ω *and* $g \in \omega^\omega$ *unbounded and monotone. Then* $(2^{\omega_1}, <_\ell)$ *embeds into* $(S_\mathcal{U}(g), <_\mathcal{U})$.

1.6 Theorem. *There is a model of* $\mathrm{MA}(\aleph_1)$ *which satisfies the statement "*$(2^{\omega_1}, <_\ell)$ *does not embed into* $(\omega^\omega, <^*)$*".*

Assuming this, the proof of 1.1 is immediate:

Proof of 1.1. Work in the model of 1.6. Assume $\neg$KC. By 1.4, there exist $\mathcal{U}$ a non-principal ultrafilter on ω, $g \in \omega^\omega$ monotone and unbounded, and an order preserving map $\pi : (S_\mathcal{U}(g), <_\mathcal{U}) \to (\omega^\omega, <)$. Since $(S_\mathcal{U}(g), <_\mathcal{U})$ is a total order, π is an order embedding. By 1.5, pick an embedding $f : (2^{\omega_1}, <_\ell) \to (S_\mathcal{U}(g), <_\mathcal{U})$. Clearly, $\pi \circ f$ is an embedding from $(2^{\omega_1}, <_\ell)$ into $(\omega^\omega, <^*)$. This contradicts 1.6. ■

2 Gaps

2.1 Definition. Let $(\mathbb{P}, <)$ be a strict order. Given $A \subseteq \mathbb{P}$ and $p \in \mathbb{P}$, we write $A < p$ iff $a < p$ for every $x \in A$. We analogously define $p < A$. Given $A, B \subseteq \mathbb{P}$, we write $A < B$ iff $\forall a \in A \forall b \in B (a < b)$.

A **pregap** is a pair (A, B) of subsets of $\mathbb{P}$ such that $A < B$. We say that a pregap (A, B) is a **gap** iff there is no $c \in \mathbb{P}$ with $A < c < B$ (such a c is said to *fill*, *interpolate* or *separate* the pregap).

A (κ, λ)**-pregap** is a pregap (A, B) such that $(A, <)$ is well-ordered with type κ and $(B, >)$ is well-ordered with type λ. Note that the order on B has been reversed. More explicitly, there exist sequences $\langle a_\alpha : \alpha < \kappa \rangle, \langle b_\beta : \beta < \lambda \rangle$ such that $A = \{a_\alpha : \alpha < \kappa\}$, $B = \{b_\beta : \beta < \lambda\}$, and for all $\alpha_1 < \alpha_2 < \kappa$ and all $\beta_1 < \beta_2 < \lambda$,

$$a_{\alpha_1} < a_{\alpha_2} < b_{\beta_2} < b_{\beta_1}.$$

A (κ, λ)**-gap** is a (κ, λ)-pregap which is also a gap. ⌟

We will often blur the distinction between the sets A, B and their enumerations. For instance, we might call the pair of enumerations a gap. This is all permissible, since (A, B) can be computed from their order preserving/reversing enumerations, and vice versa.

2.2 Lemma. *The ordered set* $(\omega^\omega, <^*)$ *has no* (ω, ω)*-gaps. Moreover, if* $(\langle f_n : n < \omega \rangle, \langle g_n : n < \omega \rangle)$ *is an* (ω, ω)*-pregap, then there exist* $f, g \in \omega^\omega$ *such that, for all* $n \in \omega$,

$$f_n <^* f \ll g <^* g_n. \tag{2}$$

Proof. Find natural numbers $k_0 < k_1 < \dots$ such that, for every $n \in \omega$ and every $i \geq k_n$,

$$f_0(i) < \cdots < f_n(i) < g_n(i) < \cdots < g_0(i).$$

Define $f(i) = f_n(i)$ whenever $k_n \leq i < k_{n+1}$ ($f(i)$ for $i < k_0$ can be defined arbitrarily) and define $g(i) = f_n(i)$ whenever $k_n \leq i < k_{n+1}$ (same comment). This works. ■

2.3 Lemma. *Assume* MA($\aleph_1$) *holds. Let* $(\langle f_\alpha : \alpha < \omega_1\rangle, \langle g_\alpha : \alpha < \omega_1\rangle)$ *be an* (ω_1, ω_1)*-pregap in* $(\omega^\omega, <^*)$. *Let* $\mathcal{U}$ *be a non-principal ultrafilter on* ω. *Then there is some* $h \in \omega^\omega$ *such that, for every* $\alpha < \omega_1$, $f_\alpha <_\mathcal{U} h <_\mathcal{U} g_\alpha$. *In other words,* $(\langle f_\alpha : \alpha < \omega_1\rangle, \langle g_\alpha : \alpha < \omega_1\rangle)$ *is not a gap in* $(\omega^\omega, <_\mathcal{U})$.

Proof. As hinted by the hypothesis, we shall define a ccc poset which introduces the desired object. First, some terminology. Given $x \subseteq \omega_1$ and $n \in \omega$, we shall say that n is large enough for x iff for every $\alpha, \beta \in x$ with $\alpha < \beta$ and every $i \geq n$,

$$f_\alpha(i) \leq f_\beta(i) \leq g_\beta(i) \leq g_\alpha(i).$$

Clearly, for every finite $x \subseteq \omega_1$ there is some $n \in \omega$ which is large enough for x. The reason to prefer $\leq$ over $<$ will become clear later.

Let $\mathbb{P}$ be the set of all triples $(x, u, v) \in [\omega_1]^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ such that $|u| = |v| =: n_p$ is large enough for x.

We remark that, if $(x, u, v) \in \mathbb{P}$ and u' and v' are (function) extensions of u and v respectively, then $(x, u', v') \in \mathbb{P}$.

Given conditions $p = (x, u, v)$ and $p' = (x', u', v')$, we say $p \leq p'$ iff the following hold

(i) $x \supseteq x'$,

(ii) $u \supseteq u'$,

(iii) $v \supseteq v'$,

(iv) for every $\alpha \in x$ and every $i \in \omega$ with $n_{p'} < i \leq n_p$, either $f_\alpha(i) \leq u(i) \leq g_\alpha(i)$ or $f_\alpha(i) \leq v(i) \leq g_\alpha(i)$.

This definition may seem rather strange (specially the disjunction in (iv)), but there's a reason the "obvious simplification" of the forcing doesn't work (see the remark after the proof). Note that $\mathbb{P}$ does not depend on $\mathcal{U}$.

Claim 1. $\mathbb{P}$ *is* σ*-linked*[2]. *In particular,* $\mathbb{P}$ *is ccc.*

Proof of claim 1. Since $\bigcup_{n\in\omega} \omega^n \times \omega^n$ is countable, it suffices to show that any two conditions which agree on their second and third coordinates are compatible. So, fix $p = (x, u, v) \in \mathbb{P}$ and $q = (y, u, v) \in \mathbb{P}$. Put $n := n_p = n_q$. By assumption, n is large enough for both x and y. Since $x \cup y$ is finite and the functions form a pregap, we can find $n' > n$ which is large enough for $x \cup y$.

Put $\delta = \max(x)$ and $\gamma = \max(y)$. Pick functions $u', v' : n' \to \omega$ which extend u and v, respectively, and such that, if $n < i \leq n'$, then $f_\delta(i) \leq u'(i) \leq g_\delta(i)$ and $f_\gamma(i) \leq v'(i) \leq g_\gamma(i)$. This is possible because n is large enough for x. Now let $r := (x \cup y, u', v')$. It is immediate that r is a condition (because n' is large enough for $x \cup y$). To see that $r \leq p$, fix $\alpha \in x$. If $n < i \leq n'$, then

$$f_\alpha(i) \leq f_\delta(i) \leq u(i) \leq g_\delta(i) \leq g_\alpha(i), \tag{3}$$

so that (iv) holds (note the relevance of the *or*). Items (i),(ii),(iii) are obvious, so $r \leq p$. The proof of $r \leq q$, is entirely analogous, replacing δ by γ in (3).

[2]Given a poset $\mathbb{Q}$, a set $L \subseteq \mathbb{Q}$ is said to be linked iff elements of L are pairwise compatible. We say $\mathbb{Q}$ is σ-linked iff it is the union of countably many linked subsets.

We remark that, if the definition of "large enough" used $<$ instead of $\leq$, then we might not be able to complete the proof. Indeed, it could happen that $f_\delta(i) + 1 = g_\delta(i)$ for some relevant[3] i, in which case we won't be able to find a natural number strictly between $f_\delta(i)$ and $g_\delta(i)$. This is the reason to prefer $\leq$. $\dashv$

We now need suitable dense sets, in order to apply $\mathrm{MA}(\aleph_1)$.

Claim 2. *Define, for $n \in \omega$ and $\alpha < \omega_1$,*

$$D_n := \{p \in \mathbb{P} : n_p \geq n\}$$

and

$$E_\alpha := \{(x, u, v) \in \mathbb{P} : \alpha \in x\}.$$

Then every D_n and every E_α is dense in $\mathbb{P}$.

Proof of claim 2. We first deal with D_n. Fix $p = (x, u, v) \in \mathbb{P}$. Suppose that $n > n_p$ (otherwise $p \in D_n$). If $\delta := \max(x)$, then $f_\delta(i) < g_\delta(i)$ for any $i \geq n_p$ because p is a condition. Extend u and v to u' and v', respectively so that $|u'| = |v'| = n$ and $f_\delta(i) < u'(i), v'(i) < g_\delta(i)$ whenever $n_p < i \leq n$. Put $q := (x, u', v')$, which is obviously in D_n. By the choice of δ, it is easy to see that $q \leq p$, and we're done. Note that we are actually verify a stronger version of (iv) (with "and" replacing "or").

Next, we deal with E_α. Fix $p = (x, u, v) \in \mathbb{P}$ and assume $\alpha \notin x$. Pick $n > |u|$ large enough for $x \cup \{\alpha\}$. If $\delta := \max(x)$ and $i \geq |u|$, then $f_\delta(i) \leq g_\delta(i)$ because $p \in \mathbb{P}$, so we can let u' and v' be any extensions of u and v, respectively, such that $|u'| = |v'| = n$ and $f_\delta(i) \leq u'(i), v'(i) \leq g_\delta(i)$. Put $q := (x \cup \{\alpha\}, u', v')$. Because $|u'| = n$, $q \in \mathbb{P}$ by choice of n. It is also easy to see that $q \in E_\alpha$ and $q \leq p$. Once again, we are verifying the stronger version of (iv). $\dashv$

By $\mathrm{MA}(\aleph_1)$, we can find a filter G on $\mathbb{P}$ which meets every member of the family $\{D_n : n \in \omega\} \cup \{E_\alpha : \alpha < \omega_1\}$. Define

$$\begin{aligned} h_1 &:= \bigcup \{u : \exists x \exists v (x, u, v) \in G\} \\ h_2 &:= \bigcup \{v : \exists x \exists u (x, u, v) \in G\}. \end{aligned}$$

Because $G \cap D_n \neq \emptyset$ for every $n < \omega$, we see that $h_1, h_2 : \omega \to \omega$. Now, fix $\alpha < \omega_1$. Because $E_{\alpha+1} \cap G \neq \emptyset$, we can find some $p = (x, u, v) \in G$ with $\alpha + 1 \in x$. Take $n \geq |u|$ such that, if $i \geq n$, then $f_\alpha(i) < f_{\alpha+1}(i) < g_{\alpha+1}(i) < g_\alpha(i)$. It now follows that, if $i \geq n$, then one of the following must hold

$$\begin{aligned} f_\alpha(i) < f_{\alpha+1}(i) \leq h_1(i) \leq g_{\alpha+1}(i) < g_\alpha(i), \\ f_\alpha(i) < f_{\alpha+1}(i) \leq h_2(i) \leq g_{\alpha+1}(i) < g_\alpha(i). \end{aligned} \tag{4}$$

Therefore, we have that, for each $i \geq n$, one of the following must hold

$$\begin{aligned} f_\alpha(i) < h_1(i) < g_\alpha(i), \\ f_\alpha(i) < h_2(i) < g_\alpha(i). \end{aligned} \tag{5}$$

[3] of course, $f_\delta \ll g_\delta$ by (the proof of) 2.2, but this only controls the eventual behaviour of the functions.

We remark that this is **NOT** the same as saying that one of the following holds

$$f_\alpha <^* h_1 <^* g_\alpha,$$
$$f_\alpha <^* h_2 <^* g_\alpha$$

There is no reason to expect that the same function (h_1 or h_2) will work for every $i \geq |u|$. In fact, this can't happen, because it is provable in ZFC that $(\omega^\omega, <^*)$ has (ω_1, ω_1)-gaps (this is due to Hausdorff). The current setting is quite Ramsey theoretic; we're colouring pairs (i, α) by two colours $\{h_1, h_2\}$. The colour depends both on i and α.

It is now, as is often the case with Ramsey theory and ω_1, that the presence of the ultrafilter comes to our aid.

Claim 3. *Either*

$$\forall \alpha < \omega_1 (f_\alpha <_{\mathcal{U}} h_1 <_{\mathcal{U}} g_\alpha)$$

or

$$\forall \alpha < \omega_1 (f_\alpha <_{\mathcal{U}} h_2 <_{\mathcal{U}} g_\alpha).$$

Proof of claim 3. Suppose not. Then, we can find $\alpha, \beta < \omega_1$ such that $\neg(f_\alpha <_{\mathcal{U}} h_1 <_{\mathcal{U}} g_\alpha)$ and $\neg(f_\beta <_{\mathcal{U}} h_2 <_{\mathcal{U}} g_\beta)$. Assume without loss of generality that $\alpha < \beta$. We know that

$$f_\alpha <^* f_\beta <^* g_\beta <^* g_\alpha,$$

hence

$$f_\alpha <_{\mathcal{U}} f_\beta <_{\mathcal{U}} g_\beta <_{\mathcal{U}} g_\alpha$$

because $\mathcal{U}$ is non-principal. This entails that $\neg(f_\beta <_{\mathcal{U}} h_1 <_{\mathcal{U}} g_\beta)$. Put

$$A := \{i \in \omega : f_\beta(i) < h_1(i) < g_\beta(i)\},$$
$$B := \{i \in \omega : f_\beta(i) < h_2(i) < g_\beta(i)\}.$$

By (5), $\omega \setminus (A \cup B)$ is finite. Since $\mathcal{U}$ is non-principal, $A \cup B \in \mathcal{U}$. Finally, since $\mathcal{U}$ is an ultrafilter, $A \in \mathcal{U}$ or $B \in \mathcal{U}$. But $A \in \mathcal{U}$ iff $f_\beta <_{\mathcal{U}} h_1 <_{\mathcal{U}} g_\beta$ and $B \in \mathcal{U}$ iff $f_\beta <_{\mathcal{U}} h_2 <_{\mathcal{U}} g_\beta$. This is a contradiction. $\dashv$

Let $h \in \{h_1, h_2\}$ witness Claim 3. Then h fills the pregap, and this completes the proof. ■

2.4 Remark. Looking back at the proof, one may be tempted to instead consider the following, seemingly simpler poset: conditions are pairs $(x, u) \in [\omega_1]^\omega \times w^{<\omega}$ with $|u|$ large enough for x. The order relation is defined by $(x, u) \leq (y, v)$ iff

- ☞ $x \supseteq y$,
- ☞ $u \supseteq v$,
- ☞ for every $\alpha \in y$ and every $i \in |u| \setminus |v|$, $f_\alpha(i) \leq u(i) \leq g_\alpha(i)$.

Call this modified poset $\mathbb{Q}$. Ostensibly, the proof cannot possibly go through with $\mathbb{Q}$, since then we would be able to fill any (ω_1, ω_1)-pregap in $(\omega^\omega, <^*)$, which is impossible by the aforementioned theorem of Hausdorff. Still, I find it instructive to pinpoint where things go wrong.

Let's try to imitate the proof of the ccc. Fix $(x, u), (y, u) \in \mathbb{Q}$. The compatibility of these two conditions is equivalent to the existence of some $v \in w^{<\omega}$ extending u such that $|v|$ is large enough for $x \cup y$ and, for every $\xi \in x \cup y$ and every $i \in [|u|, |v|)$, $f_\xi(i) \leq v(i) \leq g_\xi(i)$. But, suppose there are $\alpha \in x \setminus y, \beta \in y \setminus x$, and $j > |u|$ such that $f_\alpha(j) > g_\beta(j)$. Then there can be no such v with $j < |v|$. Of course, there are only finitely many such j, but there well could be some past $|u|$. This doesn't immediately show that the ccc fails (just that the argument we just gave doesn't work), but this can be seen indirectly as follows: suppose $\mathbb{Q}$ had the ccc. If one examines the proof of Claim 2 and defines analogous dense sets, the argument does go through, and these are dense. In particular, if (x, u) and (y, u) were compatible, they would have a common extension where the second coordinate has length above j. But we just saw that this was not possible.

By the same logic, if we switch the "or" in (iv) to an "and", the sets in Claim 2 are dense, but the poset is not ccc. ⌟

Proof of 1.5. Fix $\mathcal{U}$ and g as in the statement of 1.5. We define $\langle f_t : t \in 2^{<\omega_1} \rangle$ and $\langle g_t : t \in 2^{<\omega_1} \rangle$ with $f_t, g_t \in \omega^\omega$ and $f_t \ll g_t$. This is done by recursion on $|t|$. First, since g is unbounded and monotone, we may find $f_{\langle\rangle} \ll g_{\langle\rangle} \ll g$. Given $f_t \ll g_t$, we can reason as in 2.2 to find $f_{t^\frown 0}$ and $g_{t^\frown 1}$ with $f_t \ll f_{t^\frown 0} \ll g_{t^\frown 1} <\ll g_t$. Do this again to find $f_{t^\frown 1}$ and $g_{t^\frown 0}$ with

$$f_t \ll f_{t^\frown 0} \ll g_{t^\frown 0} \ll f_{t^\frown 1} \ll g_{t^\frown 1} \ll g_t. \tag{6}$$

If t has limit length, take $\langle \alpha_n : n < \omega \rangle$ increasing and cofinal in $\mathrm{dom}(t)$ and apply 2.2 to $\langle f_{t \restriction \alpha_n} : n < \omega \rangle$ and $\langle g_{t \restriction \alpha_n} : n < \omega \rangle$. This produces f_t, g_t such that $f_s \ll f_t \ll g_t \ll g_s$ for any $s \sqsubset t$.

Fix $b \in 2^{\omega_1}$. Then $(\langle f_{b \restriction \alpha} : \alpha < \omega_1 \rangle, \langle g_{b \restriction \alpha} : \alpha < \omega_1 \rangle)$ is an (ω_1, ω_1)-pregap in $(\omega^\omega, <_\mathcal{U})$, so by 2.3 we may find $h_b \in \omega^\omega$ such that $f_t <_\mathcal{U} h_b <_\mathcal{U} g_t$ for every $t \sqsubset b$.

Suppose $b, c \in 2^{\omega_1}$ and $b <_\ell c$, say $b \restriction \xi = c \restriction \xi$ but $b(\xi) = 0$ and $c(\xi) = 1$. Put $t := b \restriction \xi = c \restriction \xi$. Then, by (6),

$$h_b <_\mathcal{U} g_{t^\frown 0} <_\mathcal{U} f_{t^\frown 1} <_\mathcal{U} h_c.$$

This shows that $b \mapsto h_b$ is order preserving, which completes the argument. ■

Now, the statement "(A, B) is a gap" is not (upwards) absolute for models of set theory. Indeed, a gap may cease to be a gap when passing to a larger universe. For example, if we collapse ω_1 to be countable, then the pregap is filled by 2.2 applied in the extension. If we want to preserve ω_1, things are not so trivial. There is a forcing, due to Laver, which is sometimes ccc and fills a prescribed gap (for details, see section 4 of Baumgartner's chapter in [6]). On the other hand, given some class Γ of forcing posets, say that a gap (A, B) is Γ-indestructible iff for every $\mathbb{P} \in \Gamma$, $\Vdash_\mathbb{P} (A, B)$ is a gap. We will see later that, if $\mathrm{MA}(\aleph_1)$ holds, then every (ω_1, ω_1)-gap in $(\omega^\omega, <^*)$ is $\mathfrak{P}(\omega_1)$-indestructible, where $\mathfrak{P}(\omega_1)$ is the class of posets that preserve ω_1. So, in a sense, the collapsing example above is optimal.

With these absoluteness considerations in mind, we introduce the following definition:

2.5 Definition. Let $(\langle f_\alpha : \alpha < \omega_1 \rangle, \langle g_\alpha : \alpha < \omega_1 \rangle)$ be an (ω_1, ω_1)-pregap in $(\omega^\omega, <^*)$. We say that is is a **strong gap** iff

(i) for every $\alpha < \omega_1$, $f_\alpha < g_\alpha$, i.e. $f_\alpha(i) < g_\alpha(i)$ for every $i \in \omega$,

(ii) if $\alpha \neq \beta$, then there is some $i \in \omega$ with $f_\alpha(i) \geq g_\beta(i)$ or $f_\beta(i) \geq g_\alpha(i)$. ⌟

Obviously, being a strong gap is absolute for transitive models of enough set theory (with the same ω_1). Before going further, we verify that, as the name indicates:

2.6 Theorem. *Every strong gap is a gap.*

Proof. Let $(\langle f_\alpha : \alpha < \omega_1\rangle, \langle g_\alpha : \alpha < \omega_1\rangle)$ be an (ω_1, ω_1)-pregap in $(\omega^\omega, <^*)$. Suppose that there is some $h \in \omega^\omega$ with $f_\alpha <^* h <^* g_\alpha$ for every $\alpha < \omega_1$. By the Pigeonhole Principle, we can find some uncountable set $X \subseteq \omega_1$ and some fixed $n \in \omega$ such that $\forall \alpha \in X$, if $i \geq n$, then $f_\alpha(i) < h(i) < g_\alpha(i)$. By further thinning out, we may assume that the sets $\{f_\alpha \upharpoonright n : \alpha \in X\}$ and $\{g_\alpha \upharpoonright n : \alpha \in X\}$ both consist of a single element. Since X is uncountable, we may find $\alpha, \beta \in X$ with $\alpha \neq \beta$. Let $i \in \omega$, If $i < n$, then

$$f_\beta(i) = f_\alpha(i) < g_\alpha(i).$$

If $i \geq n$, then

$$f_\alpha(i) < h(i) < g_\beta(i)$$

because h fills the gap. In either case, (ii) fails, which is a contradiction. ■

2.7 Definition. Two (ω_1, ω_1)-pregaps $(\langle f_\alpha : \alpha < \omega_1\rangle, \langle g_\alpha : \alpha < \omega_1\rangle)$ and $(\langle f'_\alpha : \alpha < \omega_1\rangle, \langle g'_\alpha : \alpha < \omega_1\rangle)$ are said to be **equivalent** iff for every $\alpha < \omega_1$, $f_\alpha =^* f'_\alpha$ and $g_\alpha =^* g'_\alpha$. ⌟

The following is trivial:

2.8 Lemma. *Given two equivalent pregaps, one of them is a gap if and only if the other one is a gap.*

Looking ahead, the strategy to prove 1.6 is to show that an embedding $(2^{\omega_1}, <_\ell) \to (\omega^\omega, <^*)$ would give rise to a certain filled pregap in $(\omega^\omega, <^*)$, call it (X, Y). We will construct a finite support iteration of length ω_2 such that not only does (X, Y) appear in some intermediate extension, but it is additionally a gap there. We shall then force to seal off the gap (X, Y), in the sense that it can't become filled in any further extension. The standard bookkeeping will ensure that we capture every relevant pregap. So, if we assume the existence of an embedding in the final model, this will induce a filled pregap which must appear at some point of the construction, and hence got sealed off and can't be filled, which is a contradiction.

2.9 Lemma. *Let $(\langle f_\alpha : \alpha < \omega_1\rangle, \langle g_\alpha : \alpha < \omega_1\rangle)$ be a gap. There is a ccc notion of forcing that adds an equivalent strong gap.*

Proof. Let $\mathbb{Q}$ be the set of finite partial functions $p : \omega_1 \to \omega^\omega \times \omega^\omega$ such that

(i) if $\alpha \in \mathrm{dom}(p)$ and $p(\alpha) = (f, g)$, then $f_\alpha =^* f \leq g =^* g_\alpha$,

(ii) if $\alpha, \beta \in \mathrm{dom}(p)$ with $\alpha \neq \beta$, say $p(\alpha) = (f, g)$ and $p(\beta) = (f', g')$, then either $f \not\leq g'$ or $f' \not\leq g$.

We remark that (i) necessitates that p is injective, because $\alpha < \beta$ implies $f_\alpha < f_\beta < g_\beta < g_\alpha$. We order $\mathbb{Q}$ by $p \leq q$ iff $p \supseteq q$. Also, if $p \in \mathbb{Q}$ and s is any set with $s \subseteq q$, then $s \in \mathbb{Q}$. In particular, two conditions p and q are compatible iff $p \cup q \in \mathbb{Q}$.

Another important remark is that, for any finite $x \subseteq \omega_1$, there are only countably many $p \in \mathbb{Q}$ with $\mathrm{dom}(q) = x$. This follows from (i), since there are only countably many possible values for each $p(\alpha)$.

Claim 1. *$\mathbb{Q}$ has the ccc.*

Proof of claim 1. Suppose $A \subseteq \mathbb{Q}$ is an uncountable antichain. By the remark before the claim, we may assume, shrinking A if necessary, that $\mathrm{dom}(p) \neq \mathrm{dom}(q)$ whenever p and q distinct elements of A. By further shrinking if necessary, we may assume that $\{\mathrm{dom}(p) : p \in A\}$ forms a Δ-system with root R. Moreover, we may assume that $|R|$ is minimal among all uncountable antichains whose domains form Δ-systems. By a final shrinking using the remark preceding the claim, we may assume that $p \restriction R = q \restriction R$ for every $p, q \in A$.

Pick any two $p, q \in A$ with $p \neq q$. Put $r := p \cup q$. Since A is an antichain, $r \notin \mathbb{Q}$. On the other hand, r is finite partial function $\omega_1 \to \omega^\omega \times \omega^\omega$ and (i) holds for r, and so (ii) must fail. Pick $\alpha, \beta \in \mathrm{dom}(r)$ such that, letting $r(\alpha) = (f, g)$ and $r(\beta) = (f', g')$, $f \leq g'$ and $f' \leq g$. Because p and q are conditions, α and β can't belong to R. Without loss of generality, $\alpha \in \mathrm{dom}(p) \setminus R$ and $\beta \in \mathrm{dom}(q) \setminus R$. It follows that the conditions $p \restriction (\mathrm{dom}(p) \setminus R)$ and $q \restriction (\mathrm{dom}(q) \setminus R)$ are also incompatible.

Consider the set $\{p \restriction (\mathrm{dom}(p) \setminus R) : p \in A\}$. From the last paragraph, it is an uncountable antichain, and it trivially forms a Δ-system with root $\emptyset$. If $R \neq \emptyset$, this would contradict the minimality of $|R|$, so $R = \emptyset$ and hence $\mathrm{dom}(p) \cap \mathrm{dom}(q) = \emptyset$ whenever $p, q \in A$ and $p \neq q$.

Now let, for every $p \in A$,

$$F_p(i) := \min\{f(i) : \exists \alpha \in \mathrm{dom}(p) \exists g \in \omega^\omega (p(\alpha) = (f, g))\}$$
$$G_p(i) := \max\{g(i) : \exists \beta \in \mathrm{dom}(p) \exists f \in \omega^\omega (p(\alpha) = (f, g))\}$$

Note that, $F_p \leq G_p$ for every $p \in A$. Indeed, fix $p \in A$ and$i \in \omega$. Say $F_p(i) = f(i)$ where $p(\alpha) = (f, g)$ for some $\alpha \in \mathrm{dom}(p)$ and some $g \in \omega^\omega$. Then, by (i),

$$F_p(i) = f(i) \leq g(i) \leq G_p(i).$$

In fact, more is true. Let $p, q \in A$. Since $p \cup q \notin \mathbb{Q}$, (ii) must fail, and so we can find $\alpha \in \mathrm{dom}(p), \beta \in \mathrm{dom}(q)$, and $f, g, f', g' \in \omega^\omega$ such that $p(\alpha) = (f, g)$, $q(\beta) = (f', g')$, $f \leq g'$, and $f' \leq g$. But now, for any $i \in \omega$,

$$F_p(i) \leq f(i) \leq g'(i) \leq G_q(i).$$

In other words, $F_p \leq G_q$ for every $p, q \in A$. Therefore,

$$h(i) := \max\{F_p(i) : p \in A\} \leq G_q(i)$$

for any $q \in A$.

Fix $\alpha < \omega_1$. Since $\{\mathrm{dom}(p) : p \in A\}$ is an uncountable of non-empty and pairwise disjoint subsets of ω_1, we may find $p \in A$ such that $\alpha < \delta := \max(\mathrm{dom}(p))$. Pick $n \in \omega$ witnessing $f_\xi <^* f_\delta$ for every $\xi \in (\mathrm{dom}(p) \cup \{\alpha\}) \setminus \{\delta\}$. If $i \geq n$, then

$$h(i) = F_p(i) = f_\delta(i) > f_\alpha(i).$$

This shows that $f_\alpha <^* h$. On the other hand, if m is large enough to witness $g_\delta <^* g_\xi$ for every $\xi \in (\mathrm{dom}(p) \cup \{\alpha\}) \setminus \{\delta\}$, then, for any $I \geq m$,

$$h(i) \leq G_p(i) = g_\delta(i) < g_\alpha(i).$$

We have established that $f_\alpha <^* h <^* g_\alpha$ for every $\alpha < \omega_1$. But then $(\langle f_\alpha : \alpha < \omega_1 \rangle, \langle g_\alpha : \alpha < \omega_1 \rangle)$ is not a gap, contradiction. $\dashv$

To show the forcing assertion, start by letting

$$D_\alpha = \{p \in \mathbb{Q} : \alpha \in \mathrm{dom}(p)\}, \tag{7}$$

where $\alpha < \omega_1$. Note that D_α is dense in $\mathbb{Q}$ for every $\alpha < \omega_1$. Indeed, fix $\alpha < \omega_1$ and $q \in \mathbb{Q}$ with $\alpha \notin \mathrm{dom}(q)$. Enumerate $\mathrm{dom}(q)$ as $\{\beta_i : i < |q|\}$ and write $q(\beta) = (f^q, g^q)$. Pick $n \in \omega$ which is large enough to witness $f_\alpha <^* g_\alpha$. Define $f, g \in \omega^\omega$ as follows:

- ☞ If $i < |q|$, then choose $f(i) > g^q_{\beta_i}(i)$ and put $g(i) = f(i) + 1$,
- ☞ if $|x| \leq i < n$, let $f(i) = 0 = g(i)$,
- ☞ if $i \geq n$, the $f(i) = f_\alpha(i)$ and $g(i) = g_\alpha(i)$.

Then $f \not\leq g_\beta$ (as witnessed by i for $\beta = \beta_i$) for any $\beta \in \mathrm{dom}(q)$. We also have $f \leq g$, $f =^* f_\alpha$, and $g =^* g_\alpha$. Put $p := q \cup \{(\alpha, (f, g))\}$. By construction, p is a condition, $p \in D_\alpha$, and $p \leq q$. This shows that D_α is dense in $\mathbb{Q}$.

Fix G $\mathbb{Q}$-generic over the ground model V. Then $\bigcup G : \omega_1 \to \omega^\omega \times \omega^\omega$ because $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \omega_1$. Write $(f'_\alpha, g'_\alpha) = (\bigcup G)(\alpha)$. Conditions (i) and (ii) guarantee that $(\langle f'_\alpha : \alpha < \omega_1\rangle, \langle g'_\alpha : \alpha < \omega_1\rangle)$ is a strong gap which is equivalent to the starting gap. ■

2.10 Corollary. *Assume* $\mathrm{MA}(\aleph_1)$. *Then every* (ω_1, ω_1)*-gap in* $(\omega^\omega, <^*)$ *is* $\mathfrak{P}(\omega_1)$*-indestructible.*

Proof. Given an (ω_1, ω_1)-gap in $(\omega^\omega, <^*)$, associate $\mathbb{Q}$ to it as in the previous theorem. If G is a filter in $\mathbb{Q}$ which is $\{D_\alpha : \alpha < \omega_1\}$-generic (same definition as in (7)), then the argument at the end of the last proof still goes through. Such a G exists because we are only considering $\aleph_1$ many dense sets, $\mathrm{MA}(\aleph_1)$ holds, and $\mathbb{Q}$ has the ccc. This gives, in the universe, an equivalent strong gap. Every strong gap is $\mathfrak{P}(\omega_1)$-indestructible by the absoluteness considerations we mentioned before. Hence so is the starting gap. ■

3 Interlude on forcing

We now have all the machinery we need to prove 1.6. Before actually doing that, we recall some forcing terminology and associated facts. Following [5],

3.1 Definition. Let $\mathbb{P}$ be a forcing poset and τ a $\mathbb{P}$-name. We say that a $\mathbb{P}$-name ϑ is a **nice name for a subset of** τ iff ϑ has the form

$$\vartheta := \bigcup_{\sigma \in \mathrm{dom}(\tau)} \{\sigma\} \times A_\sigma,$$

where each A_σ is an antichain in $\mathbb{P}$. ⌟

There are not too many nice names:

3.2 Lemma. *Let* $\mathbb{P}$ *be a forcing poset and* τ *a* $\mathbb{P}$*-name. If* $\mathbb{P}$ *has the* κ*-cc, then there are at most* $|\mathbb{P}|^{\kappa|\mathrm{dom}(\tau)|}$ *many nice names for subsets of* τ*.*

Proof. There are $|\mathbb{P}|^\kappa$ many antichains in $\mathbb{P}$, and a nice name is determined by a function from $\mathrm{dom}(\tau)$ into the set of antichains of $\mathbb{P}$. ■

On the other hand, there are enough nice names to name everything of relevance:

3.3 Lemma. *Let $\mathbb{P}$ be a forcing poset and τ, μ a $\mathbb{P}$-name. There exists some ϑ which is a nice $\mathbb{P}$-name for a subset of τ and such that $\Vdash (\mu \subseteq \tau \to \mu = \vartheta)$.*

For a proof, see [5], Lemma IV.3.10.

3.4 Lemma. *Let $\mathbb{P}$ be a forcing poset that has the ccc. Suppose λ is an infinite cardinal and put $\delta := |\mathbb{P}|^{\lambda}$. Then $\Vdash 2^{\lambda} \leq \delta$ (where 2^{λ} is computed in the extension).*

This follows by counting nice names for subsets of $\check{\lambda}$, see [5], Lemma IV.3.11.

As an immediate consequence, we get the following result:

3.5 Lemma. *Assume* CH *holds. Let $\mathbb{P}$ be a forcing poset that has the ccc. Suppose $|\mathbb{P}| \leq \aleph_1$. Then $\Vdash$* CH.

In what follows, a "real" will be a function $\omega \to \omega$. Also, when dealing with names of the form $\check{x}$, we shall say "nice $\mathbb{P}$-name for a subset of x" instead of the formally more correct "nice $\mathbb{P}$-name for a subset of $\check{x}$". This is standard in modern forcing literature and will cause no confussion

3.6 Lemma. *Assume* CH *and $2^{\aleph_1} = \aleph_2$. Let $\mathbb{P}$ be a forcing poset of size $\aleph_1$ that has the ccc. There exists a set X with $|X| \leq \aleph_2$ which satisfies the following: For every $\mathbb{P}$-name $\dot{f}$ such that $\Vdash \dot{f} : \omega_1 \to \dot{\omega^{\omega}}$, there is some $\tau \in X$ with $\Vdash \tau = \dot{f}$.*

We remark that the symbol $\dot{\omega^{\omega}}$ in the statement is a $\mathbb{P}$-name for the set of reals as computed in the extension. Writing ω_1 is unambiguous by the ccc.

Given two names σ, μ, we let $\mathrm{op}(\sigma, \mu)$ be the canonical name for the ordered pair (σ, μ)

Proof. Let X be the set of all $\mathbb{P}$-names of the form $\{(\mathrm{op}(\check{\xi}, \tau_\xi), \mathbb{1}) : \xi < \omega_1\}$, where and $\langle \tau_\xi : \xi < \omega_1 \rangle$ is a sequence of nice $\mathbb{P}$-names for subsets of $\omega \times \omega$. By 3.2 and our cardinal arithmetic assumptions, there are at most

$$\aleph_1^{\aleph_1 \aleph_0} = \left(2^{\aleph_0}\right)^{\aleph_1} = \aleph_2$$

many nice $\mathbb{P}$-names for subsets of $\omega \times \omega$. But then $|X| \leq (\aleph_2)^{\aleph_1} = \aleph_2$.

To check that X has the prescribed properties, fix $\dot{f}$ such that $\Vdash \dot{f} : \omega_1 \to \dot{\omega^{\omega}}$. For each $\xi < \omega_1$, $\Vdash \dot{f}(\xi) : \omega \to \omega$, so $\Vdash \dot{f}(\xi) \subseteq \omega \times \omega$. By the Maximal Principle, choose, for each $\xi < \omega_1$, τ_ξ which is a nice $\mathbb{P}$-name for a subset of $\omega \times \omega$ such that $\Vdash \dot{f}(\xi) \subseteq \omega \times \omega \to \dot{f}(\xi) = \tau_\xi$. This implies that

$$\Vdash \tau_\xi = \dot{f}(\xi)$$

for every $\xi < \omega_1$.

Put $\tau := \{(\mathrm{op}(\check{\xi}, \tau_\xi), \mathbb{1}) : \xi < \omega_1\}$. Obviously, $\tau \in X$. Now let G be $\mathbb{P}$-generic over the ground model. Then, because $\dot{f}$ is forced to be a function,

$$\dot{f}_G = \{(\xi, \dot{f}_G(\xi)) : \xi < \omega_1\} = \{(\xi, (\tau_\xi)_G) : \xi < \omega_1\} = \{(\mathrm{op}(\check{\xi}, \tau_\xi))_G : \xi < \omega_1\} = \tau_G.$$

This shows that $\Vdash \dot{f} = \tau$. ■

Given a poset $\mathbb{P}$ and a cardinal κ, write $\mathrm{MA}_\kappa(\mathbb{P})$ for the statement "for every family $\mathscr{D}$ of dense subsets of $\mathbb{P}$ with $|\mathscr{D}| \leq \kappa$, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for every $D \in \mathscr{D}$."

The following is well known. For a proof, see Lemma V.4.2 in [5].

3.7 Proposition. $\mathrm{MA}(\aleph_1)$ *holds if and only if* $\mathrm{MA}_{\mathbb{P}}(\aleph_1)$ *holds for every ccc poset* $\mathbb{P}$ *with* $|\mathbb{P}| \le \aleph_1$.

3.8 Definition. Let $\mathbb{P}$ and $\mathbb{Q}$ be posets with $\mathbb{P} \subseteq \mathbb{Q}$ (as first order structures). We say that $\mathbb{P}$ is a **complete subposet of** $\mathbb{Q}$, written $\mathbb{P} \subseteq_c \mathbb{Q}$, iff

☞ if $X \subseteq \mathbb{P}$ is finite and has a lower bound in $\mathbb{Q}$, then X has a lower bound in $\mathbb{P}$,

☞ if $A \subseteq \mathbb{P}$ is a maximal antichain in $\mathbb{P}$, then it is also a maximal antichain in $\mathbb{Q}$. ⌟

The key property of complete subposets is that, if G is $\mathbb{P}_1$-generic over the ground model, then $G \cap \mathbb{P}_0$ is $\mathbb{P}_0$-generic over the ground model.

While the definition of complete subposet seems to be a second order statement, one can show

3.9 Lemma. *The statement* $\mathbb{P} \subseteq_c \mathbb{Q}$ *is absolute between transitive models of enough set theory.*

For a proof, see Lemma III.3.72 in [5]

3.10 Definition. Let $\varphi(\alpha, \trianglelefteq)$ denote the statement "$\alpha \in \mathrm{ON} \setminus \{0\}$ and $(\alpha, \trianglelefteq, 0)$ is a ccc forcing poset" (so, 0 is the distinguished top element).

Fix a poset $\mathbb{P}$. Write $\psi(\alpha, \dot{\trianglelefteq}, \mathbb{P})$ iff $\dot{\trianglelefteq}$ is a nice $\mathbb{P}$-name for a subset of $\alpha \times \alpha$ and $\Vdash_{\mathbb{P}} \varphi(\check{\alpha}, \dot{\trianglelefteq})$. ⌟

3.11 Lemma. *Let* $\mathbb{P}$ *be a ccc forcing poset. Let* G *be* $\mathbb{P}$*-generic over* V. *Suppose that, in* $V[G]$, $(\alpha, \le_\alpha, 0)$ *is a ccc poset, where* $\alpha \in \mathrm{ON} \setminus \{0\}$. *Then we can* $\dot{\trianglelefteq}$ *such that* $\psi(\alpha, \dot{\trianglelefteq}, \mathbb{P})$ *holds in* V *and* $\le_\alpha = (\dot{\trianglelefteq})_G$.

Proof. In $V[G]$, put $\alpha := |\mathbb{Q}|$ and let $\le_\alpha = (\tau)_G$ be such that $(\alpha, \le_\alpha, 0) \cong (\mathbb{Q}, \le_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$. Note that, in V,

$$\Vdash_{\mathbb{P}} \exists y[\varphi(\check{\alpha}, y) \wedge (\varphi(\check{\alpha}, \tau) \to y = \tau)]. \tag{8}$$

By the Maximal Principle, we can find a $\mathbb{P}$-name σ such that

$$\Vdash_{\mathbb{P}} \varphi(\check{\alpha}, \sigma) \wedge (\varphi(\check{\alpha}, \tau) \to \sigma = \tau). \tag{9}$$

By 3.3, let $\dot{\trianglelefteq}$ be a nice $\mathbb{P}$-name for a subset of $\alpha \times \alpha$ such that $\Vdash_{\mathbb{P}} (\sigma \subseteq \alpha \times \alpha \to \sigma = \dot{\trianglelefteq})$.

By (9), $\Vdash_{\mathbb{P}} \sigma \subseteq \alpha \times \alpha$, so that $\Vdash_{\mathbb{P}} \sigma = \dot{\trianglelefteq}$ and hence $\Vdash_{\mathbb{P}} \varphi(\check{\alpha}, \sigma)$. This establishes that $\psi(\alpha, \dot{\trianglelefteq}, \mathbb{P})$ holds.

By (9), $(\alpha, \dot{\trianglelefteq}_G, 0) = (\alpha, \tau_G, 0) = (\alpha, \le_\alpha, 0) \cong (\mathbb{Q}, \le_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}})$, as desired. ∎

3.12 Lemma. *Suppose that* $\mathbb{P}_1$ *is ccc,* $\mathbb{P}_0 \subseteq_c \mathbb{P}_1$, *and* $\dot{\trianglelefteq}$ *is a* $\mathbb{P}_0$*-name. Then* $\psi(\alpha, \dot{\trianglelefteq}, \mathbb{P}_1)$ *implies* $\psi(\alpha, \dot{\trianglelefteq}, \mathbb{P}_0)$.

Proof. Towards a contradiction suppose $\psi(\alpha, \dot{\trianglelefteq}, \mathbb{P}_1)$ and $\neg\psi(\alpha, \dot{\trianglelefteq}, \mathbb{P}_0)$ both hold. Fix $p \in \mathbb{P}_0$ such that $p \Vdash_{\mathbb{P}_0} \neg\varphi(\alpha, \dot{\trianglelefteq})$. Let G be $\mathbb{P}_1$-generic over V with $p \in G$. Then, letting $\trianglelefteq := (\dot{\trianglelefteq})_G = (\dot{\trianglelefteq})_{G \cap \mathbb{P}_0}$,

$$V[G] \models (\alpha, \trianglelefteq, 0) \text{ is a ccc forcing poset,} \tag{10}$$

but

$$V[G \cap \mathbb{P}_0] \models (\alpha, \trianglelefteq, 0) \text{ is not a ccc forcing poset.} \tag{11}$$

Fix $A \in V[G \cap \mathbb{P}_0]$ such that $V[G \cap \mathbb{P}_0] \models A$ is an uncountable antichain in $(\alpha, \trianglelefteq, 0)$. But A must be uncountable in $V[G]$ because $\mathbb{P}_1$ is ccc, and this contradicts (10). ∎

4 Completing the argument

Proof of 1.6. Assume GCH holds in the ground model V. Fix a bijection $\Gamma : \omega_2 \rightarrowtail\!\!\!\!\twoheadrightarrow \omega_2 \times \omega_2$ satisfying $\Gamma(\xi) = (\zeta, \mu) \to \zeta \leq \xi$. The standard Gödel pairing function used to show $\kappa \cdot \kappa = \kappa$ for an infinite cardinal κ is such a function (for $\kappa = \omega_2$).

We build a finite support iteration $\Big(\langle \mathbb{P}_\xi : \xi \leq \omega_2 \rangle, \langle \dot{\mathbb{Q}}_\xi : \xi < \omega_2 \rangle\Big)$ of length ω_2 with $\mathbb{P}_\xi$ having the ccc for every $\xi \leq \omega_2$ and $|\mathbb{P}_\xi| \leq \aleph_1$ for every $\xi < \omega_2$. The final poset $\mathbb{P}_{\omega_2}$ will force $\mathrm{MA}(\aleph_1)$ and "$(2^{\omega_1}, <_\ell)$ does not embed into $(\omega^\omega, <^*)$".

Consider the sets

$$\mathcal{E} := \{\gamma + 2n : n \in \omega \wedge \gamma < \omega \text{ is a limit}\},$$
$$\mathcal{O} := \omega_2 \setminus \mathcal{E}$$

of even and odd ordinals below w_2, respectively.

As we construct the iteration, once we have built $\mathbb{P}_\zeta$, for $\zeta < \omega_2$, we let $\langle (\alpha^\mu_\zeta, \dot{\trianglelefteq}^\mu) : \mu \in \mathcal{E} \rangle$ list all pairs $(\alpha, \dot{\trianglelefteq})$ such that $0 < \alpha < \omega_2$ and $\dot{\trianglelefteq}$ is a nice $\mathbb{P}_\zeta$-name for a subset of $\alpha \times \alpha$. This is possible because GCH holds in V and $|\mathbb{P}_\zeta| \leq \aleph_1$, so 3.2 shows there are at most $\aleph_1^{\aleph_1} = \aleph_2$ many such names.

We also need to list another collection of relevant names. Let $D \subseteq 2^{\omega_1}$ be the set of eventually constant sequences (this is all happening in V). More precisely, $x \in D$ iff there exist $\alpha < \omega_1$ and $i < 2$ such that $x(\xi) = 2$ for all $\xi \geq \alpha$. Note that D is $<_\ell$ dense in 2^{ω_1}, in the sense that for all $a, b \in 2^{\omega_1}$ with $a <_\ell b$ there is some $d \in D$ with $a <_\ell d <_\ell b$. By GCH, $|D| = \aleph_1$. Now, suppose $\dot{\pi}$ is a $\mathbb{P}_\zeta$-name for a function $\dot{\pi} : D \to \dot{\omega^\omega}$. Applying 3.6 (with some enumeration of D in type ω_1), we can find a sequence $\langle \dot{\pi}^\mu_\zeta : \mu \in \mathcal{O} \rangle$ of $\mathbb{P}_\zeta$-names such that, if $\dot{\pi}$ is any $\mathbb{P}_\zeta$-name for a map from D into the reals (of the extension), then $\Vdash_{\mathbb{P}_\zeta} \dot{\pi} = \dot{\pi}^\mu_\zeta$ for some $\mu \in \mathcal{O}$.

Since the iteration has finite supports, it suffices to specify what to do at successor stages. So, fix $\xi < \omega_2$. Say $(\zeta, \mu) = \Gamma(\xi)$.

If $\mu \in \mathcal{E}$, we let $\dot{\mathbb{Q}}_\xi$ be a $\mathbb{P}_\xi$-name for the trivial poset, *unless* it just so happens that $\Vdash_{\mathbb{P}_\xi} \psi(\alpha^\mu_\zeta, \dot{\trianglelefteq}^\mu_\zeta, \check{0})$. In that case, we let $\dot{\mathbb{Q}}_\xi$ be $(\alpha^\mu_\zeta, \dot{\trianglelefteq}^\mu_\zeta, \check{0})$. To be completely formal, we should write $(\dot{\mathbb{Q}}_\xi, \dot{\preceq}_{\mathbb{Q}_\xi}, \mathbb{1}_{\mathbb{Q}_\xi}) = (\alpha^\mu_\zeta, \dot{\trianglelefteq}^\mu_\zeta, \check{0})$, but we will avoid doing that.

If $\mu \in \mathcal{O}$, we let $\dot{\mathbb{Q}}_\xi$ be a $\mathbb{P}_\xi$-name for the trivial poset, *unless* it just so happens that

$$\Vdash_{\mathbb{P}_\zeta} \dot{\pi}^\mu_\zeta : (D, <_\ell) \to (\dot{\omega^\omega}, <^*) \text{ is order preserving.} \tag{12}$$

In that case, we proceed as follows. Let G_ξ be $\mathbb{P}_\xi$ generic over V and put $\pi := (\dot{\pi}^\mu_\zeta)_{G_\xi}$. Since $\mathbb{P}_\xi$, $\omega_1^V = \omega_1^{V[G_\xi]}$, so we'll just write ω_1.

Let $X := (2^{\omega_1})^V$. For each $b \in X \setminus D$, pick $\langle d^b_\alpha : \alpha < \omega_1 \rangle$ and $\langle e^b_\alpha : \alpha < \omega_1 \rangle$ which are $<_\ell$ increasing and decreasing, respectively, and such that

(I) for all $\alpha < \omega_1$, $d^b_\alpha, e^b_\alpha \in D$,

(II) for all $\alpha < \omega_1$, $d^b_\alpha <_\ell b <_\ell e^b_\alpha$,

(III) $\langle d^b_\alpha : \alpha < \omega_1 \rangle$ and $\langle e^b_\alpha : \alpha < \omega_1 \rangle$ both converge to b (as nets in order topology on $(X, <_\ell)$).

Put $f^b_\alpha := \pi(d^b_\alpha)$ and $g^b_\alpha := \pi(e^b_\alpha)$. Obviously, $(\langle f^b_\alpha : \alpha < \omega_1 \rangle, \langle g^b_\alpha : \alpha < \omega_1 \rangle)$ form an (ω_1, ω_1)-pregap in $(\omega^\omega, <^*)$ (in $V[G_\xi]$). The key point is that they cannot all be filled:

Claim 1. *There is a $b \in X \setminus D$ such that $(\langle f^b_\alpha : \alpha < \omega_1\rangle, \langle g^b_\alpha : \alpha < \omega_1\rangle)$ is a gap.*

Proof of claim 1. Suppose otherwise. For every $b \in X \setminus D$, pick $h_b \in \omega^\omega$ such that $f^b_\alpha <^* h_b <^* g^b_\alpha$ for every $\alpha < \omega_1$. Now suppose $b, c \in X \setminus D$ are such that $b < c$. By density, find $d \in D$ such that $b <_\ell d <_\ell c$. By (III), pick $\alpha < \omega_1$ such that

$$b <_\ell e^b_\alpha <_\ell d <_\ell d^c_\alpha <_\ell c.$$

Then

$$h_b <^* g^b_\alpha <^* \pi(d) <^* f^c_\alpha <^* h_c.$$

This shows that the map $X \setminus D \to \omega^\omega$ defines by $b \mapsto h_b$ is injective. In particular, $(|X \setminus D| \leq |\omega^\omega|)^{V[G_\xi]}$.

On the other hand, V, $X \setminus D$ has size $\aleph_2$, because GCH holds. But $\mathbb{P}_\xi$ is ccc, so $(|X \setminus D| = \aleph_2)^{V[G_\xi]}$. On the other hand, by 3.5, CH holds in $V[G_\xi]$, so $(|\omega^\omega| = \aleph_1)^{V[G_\xi]}$. But then the previous paragraph yields $\aleph_2 < \aleph_1$, contradiction. $\dashv$

Fix $b \in X \setminus D$ such that $(\langle f^b_\alpha : \alpha < \omega_1\rangle, \langle g^b_\alpha : \alpha < \omega_1\rangle)$ is a gap. Let $\mathbb{Q}$ be the associated ccc forcing as described in 2.9. Now step out back to V, and let $\mathbb{Q}_\xi$ be a $\mathbb{P}_\xi$-name for this poset.

This completes the construction of the iteration. We now show that $\mathbb{P}_{\omega_2}$ forces MA($\aleph_1$)+"$(2^{\omega_1}, <_\ell)$ does not embed into $(\omega^\omega, <^*)$". Being a finite support iteration of ccc posets, $\mathbb{P}$ is ccc, thus all cardinals and cofinalities are preserved. Fix an enumeration $D = \{d_\alpha : \alpha < \omega_1\}$. Write $\mathbb{P} := \mathbb{P}_{\omega_2}$ to simplify the notation.

Let G be $\mathbb{P}$-generic over V. We first show that $V[G] \models (2^{\omega_1}, <_\ell)$ does not embed into $(\omega^\omega, <^*)$. Working in $V[G]$, let $\pi : (D, <_\ell) \to (\omega^\omega, <^*)$ be an order preserving map, say $\pi = \dot{\pi}_G$. For each $\alpha < \omega_1$, let $\dot{f}_\alpha$ be a nice $\mathbb{P}$-name for a subset of $\omega \times \omega$ such that $\pi(\alpha) = \dot{f}_\alpha$. Since $\mathbb{P}$ is ccc, $\dot{f}_\alpha$ (or rather its transitive closure) mentions at most $\aleph_1$ many conditions in $\mathbb{P}$. We can therefore find $\zeta < \omega_2$ such that all the f_α's are $\mathbb{P}_\zeta$-names (for subsets of $\omega \times \omega$). Let

$$\tau := \{((\mathrm{op}(\check{\alpha}, \dot{f}_\alpha)), \mathbb{1}) : \alpha < \omega_1\}$$

Obviously, τ is a $\mathbb{P}_\zeta$-name. Also, $\Vdash_{\mathbb{P}_\xi} \tau = \dot{\pi}$ (and $\Vdash_{\mathbb{P}} \tau = \dot{\pi}$).

Claim 2. $\Vdash_{\mathbb{P}_\zeta} \tau : (\check{D}, <_\ell) \to (\dot{\omega^\omega}, <^*)$ *is order preserving*

Proof of claim 2. Let G_ζ be $\mathbb{P}_\zeta$ generic over V and let H be $\mathbb{R}$-generic over $V[G_\zeta]$, where $\mathbb{R}$ is the factor poset $\mathbb{P}/G_\zeta$. Put $G = G_\zeta * H$, which must be $\mathbb{P}$-generic over V. In $V[G]$, τ_G is an order preserving map from $(D, <_\ell)$ to $((\omega^\omega)^{V[G]}, <^*)$, because $\Vdash_{\mathbb{P}} \tau = \dot{\pi}$. But the $\dot{f}_\alpha$'s are $\mathbb{P}_\zeta$ names, so $(\dot{f}_\alpha)_G = (\dot{f}_\alpha)_{G_\zeta} \in (\omega^\omega)^{V[G_\zeta]}$. So, $\tau_{G_\zeta} = \tau_G : D \to (\omega^\omega)^{V[G_\zeta]}$ and it must preserve the relevant orderings because that is an absolute statement that holds in $V[G]$. $\dashv$

By construction, we can find $\mu \in \mathcal{O}$ such that $\Vdash_{\mathbb{P}_\zeta} \tau = \pi^\mu_\zeta$. Let $\xi < \omega_2$ be such that $\Gamma(\xi) = (\zeta, \mu)$. In $V[G \cap \mathbb{P}_\zeta]$, there is some $b \in X \setminus D$ such that

$$(\langle f^b_\alpha : \alpha < \omega_1\rangle, \langle g^b_\alpha : \alpha < \omega_1\rangle)$$

is a gap. By Claim 2 and the definition of $\mathbb{Q}_{\xi+1}$,

$$V[G \cap \mathbb{P}_{\xi+1}] \models (\langle f^b_\alpha : \alpha < \omega_1\rangle, \langle g^b_\alpha : \alpha < \omega_1\rangle) \text{ is a strong gap.}$$

But being a strong gap is absolute, so

$$V[G] \models (\langle f^b_\alpha : \alpha < \omega_1 \rangle, \langle g^b_\alpha : \alpha < \omega_1 \rangle) \text{ is a strong gap.} \tag{13}$$

Staying in $V[G]$, this shows that π cannot be extended to an order preserving map $(2^{\omega_1}, <_\ell) \to (\omega^\omega, <^*)$. Indeed, assume there was some extension π^*. Then, for any $\alpha < \omega_1$, $d^b_\alpha <_\ell b <_\ell e^b_\alpha$, so

$$f^b_\alpha <^* \pi^*(b) <^* g^b_\alpha.$$

But then $\pi^*(b)$ fills the gap $(\langle f^b_\alpha : \alpha < \omega_1 \rangle, \langle g^b_\alpha : \alpha < \omega_1 \rangle)$, contradiction.

Taking stock, we've shown that, in $V[G]$, no order preserving map $\pi : (D, <_\ell) \to (\omega^\omega, <^*)$ can be extended to $(2^{\omega_1}, <_\ell) \to (\omega^\omega, <^*)$. But this means there can be no order preserving map $(2^{\omega_1}, <_\ell) \to (\omega^\omega, <^*)$ at all in $V[G]$, as was claimed.

The proof that $V[G] \models \mathrm{MA}(\aleph_1)$ is very much the same as the usual proof of the consistency of $\mathrm{MA}(\aleph_1)$. We include it here for the reader's convenience.

Work in $V[G]$. By 3.7, it suffices to show that $\mathrm{MA}(\aleph_1)(\mathbb{Q})$ holds for every ccc $\mathbb{Q}$ such that $|\mathbb{Q}| \leq \aleph_1$. Fix such a $\mathbb{Q}$ and assume without loss of generality that $\mathbb{Q}$ is of the form $(\alpha, \trianglelefteq, 0)$ for some $\alpha < \omega_1$. Using 3.11, find $\dot{\trianglelefteq}$ such that $\trianglelefteq = \dot{\trianglelefteq}_G$ and $\psi^V(\alpha, \dot{\trianglelefteq}, \mathbb{P})$. In particular, $\dot{\trianglelefteq}$ is a nice $\mathbb{P}$-name for a subset of $\alpha \times \alpha$.

Fix a sequence $\langle D_\delta : \delta < \omega_1 \rangle$ of dense subsets of $(\alpha, \trianglelefteq, 0)$. For each $\delta < \omega_1$, let $\dot{D}_\delta$ be a nice $\mathbb{P}$-name for a subset of α such that $(\dot{D}_\delta)_G = D_\delta$. As before, we are considering $\aleph_1$ many nice names, so we can find some $\zeta < \mu$ such that $\dot{\trianglelefteq}$ and all the $\dot{D}_\delta$ are $\mathbb{P}_\zeta$-names. Fix $\mu \in \mathcal{E}$ such that $(\alpha, \dot{\trianglelefteq}) = (\alpha^\mu_\zeta, \dot{\trianglelefteq}^\mu)$. By 3.12, $\psi(\alpha, \dot{\trianglelefteq}, \mathbb{P})$ implies $\psi(\alpha^\mu_\zeta, \dot{\trianglelefteq}^\mu, \mathbb{P}_\zeta)$. But then $\dot{\mathbb{Q}}_\xi$ is $(\alpha, \dot{\trianglelefteq}, 0)$, and so $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi * (\alpha, \dot{\trianglelefteq}, 0)$. If we accordingly factor $G \cap \mathbb{P}_{\xi+1}$ as $(G \cap \mathbb{P}_\xi) * H$, then $H \in V[G \cap \mathbb{P}_{\xi+1}] \subseteq V[G]$ is $\mathbb{Q}$-generic over $V[G \cap \mathbb{P}_\xi]$. But $D_\delta \in V[G \cap \mathbb{P}_\xi]$ for every $\delta < \omega_1$, and so H meets every D_δ. This shows that $\mathrm{MA}(\aleph_1)(\mathbb{Q})$ holds, and completes the argument. ■

5 Further results and questions

The question of whether KC implies CH was settled in the negative:

5.1 Theorem (Woodin, 1993). *Assume* CH. *Let $\mathbb{P}$ be the usual forcing for adding ω_2 many Cohen reals. Then $\Vdash_{\mathbb{P}} \neg\mathrm{KC}$.*

For a proof, see [8]. Higher values of $2^{\aleph_0}$ are also compatible with KC, see [3].

Strengthening $\mathrm{MA}(\aleph_1)$, we get a direct implication instead of the consistency result 1.1.

5.2 Theorem (Todorcevic). PFA *implies* KC.

This follows from Theorem 8.8 on page 76 of [7].

In [8], it is said that whether $\mathrm{MA}(\aleph_1)$ implies KC is open. As far as I know, this is still the case.

Going in a different direction, let $\mathfrak{X}$ be the class of infinite compact Hausdorff spaces. Let $\varphi(X)$ be the statement "$X \in \mathfrak{X}$ and, for every Banach algebra A, every algebra homomorphism $C(X) \to A$ is continuous". Is the theory

$$\mathrm{ZFC} + \exists X, Y \in \mathfrak{X}[\varphi(X) \wedge \neg\varphi(Y)].$$

consistent?

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