Finitely branching subtrees of Aronszajn trees

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Preliminaries on trees

A *tree* is a pair $(T, <_T)$ such that $<_T$ is a strict partial order on T and $\{y \in T : y <_T x\}$ is well-ordered for every $x \in T$. The *height* of $x \in T$, is the order-type of $\{y \in T : y <_T x\}$, and is denoted $\mathrm{ht}_T(x)$. The set $T_\alpha = \{x \in T : \mathrm{ht}_T(x) = \alpha\}$ is the *αth-level* of T.

Given $x \in T$, $I(x)$ is the set of *immediate successors* of x. A tree T is *finitely branching* if $|I(x)| < \aleph_0$ for every $x \in T$, and *infinitely branching* if $|I(x)| \geq \aleph_0$ for every $x \in T$.

A *chain* in a tree T is a subset of T which is linearly ordered by $<_T$. A *branch* is a maximal chain. A *cofinal branch* is a branch which meets every level of T.

A tree is said to be an *$\aleph_1$-tree* if every level is countable and its height is $\aleph_1$.

An tree is *Aronszajn* if it is an $\aleph_1$ tree with no cofinal branches.

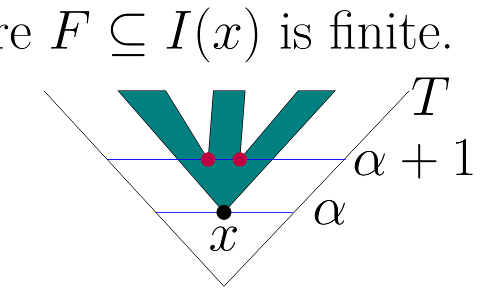
An *antichain* in a tree is a set of pairwise incomparable elements of T. A tree is *Suslin* if it is an Aronszajn tree with no uncountable antichains.

A *subtree* of a tree is a downards closed subset. An infinitely branching tree is *Lindelöf* if it is an $\aleph_1$-tree with no uncountable finitely branching subtrees.

A topology on trees

Given a tree T, Nyikos defines the *fine-wedge topology* on T to be generated by all sets of the form $\uparrow x$ and their complements, where $x \in T$ and $\uparrow x = \{y \in T : x \leq y\}$.

A local base at x is given by all sets of the form $\uparrow x \setminus \uparrow F$, where $F \subseteq I(x)$ is finite.



The topology is Hausdorff (and more).

If we regard points on a tree as being "trivial" whenever $|I(x)| < \aleph_0$, then being trivial in the tree sense is the same as being trivial topologically (i.e. isolated).

Trees as Lindelöf spaces

Basic open covers of a tree T can be coded by functions f with domain T and $f(x) \in [I(x)]^{<\omega}$, namely

$$\mathcal{U}_f = \{\uparrow x \setminus \uparrow f(x) : x \in T\}.$$

The cover $\mathcal{U}_f$ has a countable subcover if and only if $\mathcal{U}_{f\restriction(T\restriction\alpha)}$ covers T for some $\alpha < \omega_1$.

A point $x \in T$ is *safe* if for all $y < x$, $x \in \uparrow f(y)$.

A safe point at level α is a witness to $\mathcal{U}_{f\restriction(T\restriction\alpha)}$ not covering T.

The set of safe points is a finitely branching subtree of T.

Theorem

An infinitely branching $\aleph_1$-tree is Lindelöf if and only if it has the Lindelöf property with respect to the fine wedge topology.

($\diamondsuit$) Special Lindelöf trees

An $\aleph_1$-tree is *special* if it can be written as a countable union of antichains. Clearly, no Suslin tree is special. On the other hand, it is consistent to have a special Lindelöf tree.

Assume $\diamondsuit$ holds. Use the $\diamondsuit$-sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ to construct an infinitely branching normal tree T by recursion on levels, ensuring that S_α never grows to be an uncountable finitely branching subtree of T. In parallel, construct an order preserving function $T \to \mathbb{Q}$. The resulting tree is special and non-Lindelöf.

At successor stages $\alpha = \beta + 1$, put $\aleph_0$ new nodes immediately above every node on level β.

At limit stages α, for each $x \in T\restriction\alpha$ build a cofinal branch b_x through $T\restriction\alpha$ such that $x \in b_x$ and, if $x \in S_\alpha$ and S_α is finitely branching, then $b_x \not\subseteq S_\alpha$. This is possible by the construction at successor stages. Put a new node above b_x on level α. This seals-off S_α.

Locating the class of Lindelöf trees

Branches can be regarded as 1-branching subtrees of a tree, so Lindelöf trees are Aronszajn.

Suppose T is an infinitely branching Suslin tree. Let S be an uncountable, finitely branching subtree of T. Then S is also Suslin, and forcing with S adds a cofinal branch through T, say b.

By general facts, b is generic for T over V.

Since T is infinitely branching and S is finitely branching, $T \setminus S$ is dense in T. So $b \cap T \setminus S \neq \emptyset$. But $b \subseteq S$, contradiction.

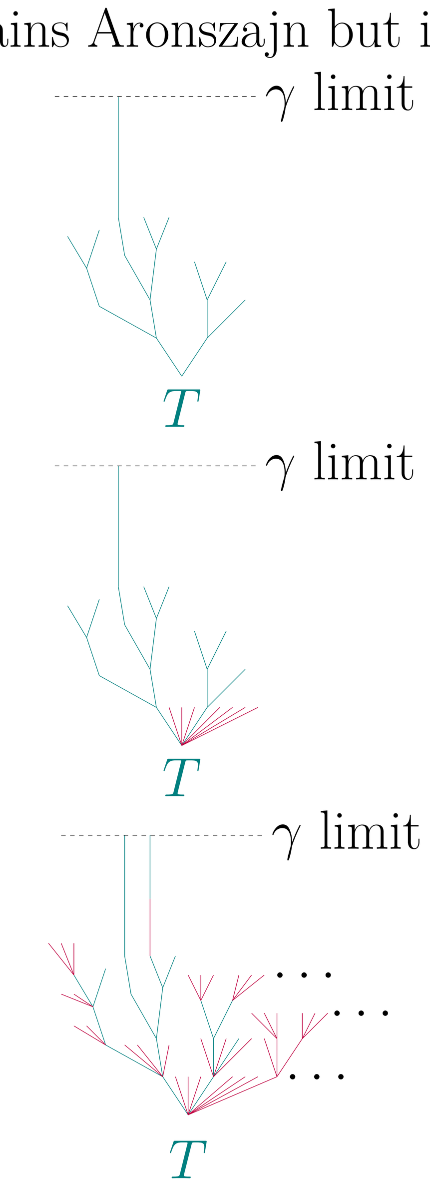
Theorem

Suslin $\subseteq$ Lindelöf $\subsetneq$ Aronszajn,

A non-Lindelöf Aronszajn tree

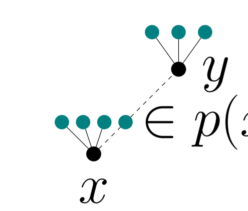
Fix a coherent sequence of injections $e_\alpha : \alpha \to \omega$. Use a bijection $\omega_1 \times \omega \to \omega$ to obtain a coherent sequence $\langle f[e_\alpha] : \alpha < \omega_1 \rangle$. Use this sequence to build an Aronszajn subtree of ${}^{<\omega_1}2$, call it T.

Recursively grow new nodes at every point of T. The outer tree remains Aronszajn but it not Lindelöf.



Forcing tall subtrees

Let T be an infinitely branching $\aleph_1$-tree. Let $\mathbb{P}_T$ be the poset of finite partial functions with $\mathrm{dom}(p) \subseteq T$, $p(x) \in [I(x)]^{<\omega} \setminus \{\emptyset\}$ for every $x \in \mathrm{dom}(p)$ and, if $x, y \in \mathrm{dom}(p)$ and $x < y$, then $y\restriction(\mathrm{ht}(x)+1) \in p(x)$.



Intuitively, p is a promise that, for each $x \in \mathrm{dom}(p)$, the set of immediate successors of x in the generic subtree is $p(x)$.

Let $\dot{S} = \{\mathrm{dom}(p) : \exists p \in \dot{G}(x \in \mathrm{dom}(p))\}$, where $\dot{G}$ is a $\mathbb{P}_T$-name for the generic filter.

Theorem

Let T be an infinitely branching normal Aronszajn tree. Then $\mathbb{P}_T$ is ccc and $\Vdash_{\mathbb{P}_T}$ "$\dot{S}$ is a finitely branching normal subtree of T". Also, T remains Aronszajn in the $\mathbb{P}_T$-extension.

The poset $\mathbb{P}_T$ is merely ccc, (consistently) not Knaster, so why doesn't it add branches?

Let $\mathbb{S}_T$ be Baumgartner's poset for specializing T with finite conditions.. Both $\mathbb{P}_T$ and $\mathbb{S}_T$ are absolute.

Suppose G is $\mathbb{P}$-generic over V and T has a cofinal branch in $V[G]$. Then the same is true in $V[G][H]$, where H is $\mathbb{S}_T$-generic over $V[G]$. But $V[G][H] = V[H][G]$ and, in $V[H]$, T is special, hence remains special in $V[G][H]$, contradiction.

Theorem

If $\mathsf{MA}_{\aleph_1}$-holds, then there are no Lindelöf trees.

Other ways of adding subtrees?

One can show that if a poset is either countably closed or strongly proper, then it cannot add a new uncountable finitely branching subtree. Both of these proofs use the topological characterization of Lindelöfness.